

ON A REGULARIZED FAMILY OF MODELS FOR THE FULL ERICKSEN-LESLIE SYSTEM

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ABSTRACT. We consider a general family of regularized systems for the full Ericksen-Leslie model for the hydrodynamics of liquid crystals in n -dimensional compact Riemannian manifolds. The system we consider consists of a regularized family of Navier-Stokes equations (including the Navier-Stokes- α -like equation, the Leray- α equation, the Modified Leray- α equation, the Simplified Bardina model, the Navier-Stokes-Voigt model and the Navier-Stokes equation) for the fluid velocity u suitably coupled with a parabolic equation for the director field d . We establish existence, stability and regularity results for this family. We also show the existence of a finite dimensional global attractor for our general model, and then establish sufficiently general conditions under which each trajectory converges to a single equilibrium by means of a Łojasiewicz-Simon inequality.

1. INTRODUCTION

A nematic liquid crystal is a phase of a material between the solid and liquid phases, with the liquid phase having a certain degree of orientational order. The flow in the liquid phase is described by a velocity $u = (u_1, \dots, u_n)$ and by a director field $d = (d_1, \dots, d_n)$, which stands for the averaged macroscopic/continuum orientation in \mathbb{R}^n of the constituent molecules. One model that governs the flow of the nematic liquid crystals is the general Ericksen-Leslie system (NS-EL) with Ginzburg-Landau type approximation proposed in [22]. This system consists of the Navier-Stokes equation for the fluid velocity coupled with two additional anisotropic stress tensors, which are the elastic (Ericksen) and the viscous (Leslie) stress tensors, respectively, and a parabolic equation for the director field. Among the mathematical rigorous results for the full (NS-EL) system one can barely find the references [8, 22] for incompressible fluid flows. These contributions are mainly concerned with well-posedness and long-time behavior of solutions to the system under suitable assumptions on the Leslie coefficients, ensuring that a certain natural energy associated with the (NS-EL) system is dissipated. Especially in [8], existence of a global-in-time weak solutions with *finite energy* is proved as well as blow-up criterion is developed for the existence of a globally-well defined classical solution of the 3D (NS-EL) system with periodic boundary conditions. On the other hand, in [22] global well-posedness of smooth solutions is established in certain special cases and Lyapunov stability for this system near local energy minimizers is shown. Due to the highly nonlinear and strong coupling of the (NS-EL) system most of previous analytical studies were always restricted to some simplified versions of the (NS-EL) system. Rather than giving a full account of the literature, we refer the reader to [8] where a complete description of the most up-to-date analytical studies has been undertaken in detail for these simplified models.

Regularized flow equations in hydrodynamics play a key role in understanding turbulent phenomena in science. Given the nonlinear nature of turbulent nematic liquid crystal flows and the ensuing multiscale interactions, direct numerical simulations of the turbulent nematic liquid crystal flows is still presently lacking apart from some investigations performed on simplified systems which still retain the basic nonlinear structure and the essential features of the full hydrodynamic (NS-EL) equations [2, 3, 17]. This is due mainly to two factors: (a) the numerical computation of the 3D Navier-Stokes (NSE) equation with a high Reynolds number in regimes in which the nonlinearities prevail is not possible at present [9] and (b) the strong coupling in the Ericksen-Leslie (NS-EL) equations make the numerical approximation and computation of the solution quite expensive for the nowadays computers, even for simplified versions of the original system [3]. Indeed, in turbulent flows most of the computational difficulties lie in the understanding the

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dynamic interaction between small and large scales of the flow [19]. Moreover, the number of degrees of freedom needed to simulate the fluid flow increases quite drastically as a function of the Reynolds number. In order to overcome these issues, in recent years the approach of regularization modeling has been proposed and tested successfully against experimental data for the 3D NSE equation. One novelty of this approach is that the regularization models of the 3D Navier-Stokes equation only modifies the spectral distribution of energy, and the well-posedness (i.e., existence, uniqueness and stability with respect to the initial data) of solutions can be rigorously proven unlike for the 3D (NSE) equation [14]. In order to handle these problems for a simplified model of the original Ericksen-Leslie system, a general three-parameter family of regularized equations has been proposed and investigated in [11] for the purpose of direct numerical simulations of turbulent incompressible flows of nematic liquid crystals. Existence and uniqueness of smooth solutions can be rigorously proven for the regularized family of [11, Section 7], as well as the existence of finite dimensional global attractors and, under proper natural conditions, the eventual asymptotic stabilization of the corresponding solutions to single equilibria. The robust analytical properties of these simplified Ericksen-Leslie models ensure computability of their solutions and the stability of numerical schemes.

In this paper, our main goal is to investigate a wide range of regularized models for the general Ericksen-Leslie system (NS-EL) with Ginzburg-Landau type approximation proposed in [22]. As in [11], we will mainly be concerned with the same fundamental issues in the theory of infinite-dimensional dynamical systems, that to give a unified analysis of the entire family of regularized models and to establish existence, stability and regularity results, and long-time results. As a representative of a more general model, described in detail in the next section, the family of regularized models associated with the original (NS-EL) system we wish to consider formally reads

$$(1.1) \quad \begin{cases} \partial_t u + A_0 u + (Mu \cdot \nabla)(v) + \chi \nabla (Mu)^T \cdot (v) + \nabla p = -\operatorname{div}(\nabla d \odot \nabla d - \sigma_Q) + g, \\ \partial_t d + v \cdot \nabla d - \omega_Q d + \frac{\lambda_2}{\lambda_1} A_Q d = \frac{1}{\lambda_1} (A_1 d + \nabla_d W(d)), \\ u = Q^{-1} v, \\ \operatorname{div}(u) = 0, \operatorname{div}(v) = 0, \\ u(0) = u_0, \\ d(0) = d_0. \end{cases}$$

Here, A_0 , A_1 , M , and Q are linear operators having certain mapping properties and χ is either 1 or 0. The function $W(d) = (|d|^2 - 1)^2$ is used as a typical approximation to penalize the deviation of the length $|d|$ from the value 1, under a generally accepted assumption that the liquid crystal molecules are of similar size [22]. Following [14] (cf. also [11]), there are three parameters which control the degree of smoothing in the operators A_0 , M and Q , namely θ , θ_1 and θ_2 , while A_1 is a differential operator of *second* order. Some examples of operators A_0 , A_1 , M , and Q which satisfy the mapping assumptions imposed in this paper are

$$(1.2) \quad A_0 = \mu_4(-\Delta)^\theta, \quad A_1 = -\Delta, \quad M = (I - \alpha^2 \Delta)^{-\theta_1}, \quad Q = (I - \alpha^2 \Delta)^{-\theta_2},$$

for fixed positive real numbers α, μ_4 and for specific choices of the real parameters θ , θ_1 , and θ_2 . The notation ∇_d represents the gradient with respect to the variable d . Besides, the term $\nabla d \odot \nabla d$ denotes the $n \times n$ matrix whose (i, j) -th entry is given by $\nabla_i d \cdot \nabla_j d$, for $1 \leq i, j \leq n$, while for $v := Qu$,

$$(1.3) \quad A_Q = \frac{1}{2}(\nabla v + \nabla^T v), \quad \omega_Q = \frac{1}{2}(\nabla v - \nabla^T v),$$

represent the rate of strain tensor and the skew-symmetric part of the strain rate, respectively. Moreover, as in [8, 22] we denote by

$$(1.4) \quad \dot{d} = \partial_t d + v \cdot \nabla d, \quad \mathcal{N}_Q = \dot{d} - \omega_Q d$$

the material derivative of d and the rigid rotation part of the changing rate of the director by fluid vorticity. The kinematic transport $\lambda_1 \mathcal{N}_Q + \lambda_2 A_Q d$ represents the effect of the macroscopic flow field on the microscopic structure such that the material coefficients λ_1 and λ_2 reflect the molecular shape and how slipper the particles are in the fluid, respectively. The Leslie stress tensor σ_Q takes on the following general form:

$$(1.5) \quad \sigma_Q = \mu_1(d^T A_Q d)d \otimes d + \mu_2 \mathcal{N}_Q \otimes d + \mu_3 d \otimes \mathcal{N}_Q + \mu_5(A_Q d) \otimes d + \mu_6 d \otimes (A_Q d),$$

where \otimes stands for the usual Kronecker product, i.e., $(a \otimes b)_{ij} := a_i b_j$, for $1 \leq i, j \leq n$. The six independent coefficients μ_1, \dots, μ_6 from (1.2) and (1.5), are called Leslie coefficients. Finally, $g = g(t)$ is an external body force acting on the fluid.

It is rather clear that one recovers the original (NS-EL) system of [8, 22] by setting $\theta = 1$, $\theta_1 = \theta_2 = 0$ and $\chi = 0$ in (1.1). We recall that some theoretical aspects (i.e., existence of globally-defined weak solutions and blow-up criteria for smooth solutions) have been recently developed in [8]. Beyond [8] and [22], not much else seems to be known in terms of analytical and numerical results for this system to the best of our knowledge. It is worth noting that among the models considered in (1.1)-(1.2), when restricted only to the equation for the fluid velocity, one can also find the globally well-posed 3D Leray- α equations, the modified 3D Leray (ML) equations, the simplified 3D Bardina (SBM) models, the 3D Navier-Stokes-Voigt (NSV) equations, and their inviscid counterparts. The corresponding parameter values of $(\theta, \theta_1, \theta_2)$ and operators (1.2) associated with these models are described in detail in Section 2. Inspired by work in [11] performed on a simplified family of Ericksen-Leslie models, we proceed to develop a complete theory for the whole family of (1.1). First, we develop well-posedness and long-time dynamics results for the entire three-parameter family of models (1.1), and then subsequently recover results of this type for the specific regularization models that have not been previously studied in the literature, including results for the original (NS-EL) system.

The main novelties of the present paper with respect to previous results on the original (NS-EL) model are the following:

(i) The existence result of the globally-defined weak solutions with *finite* energy are extended to the general family of models (1.1) on an n -dimensional compact Riemannian manifold with or without boundary. We also address both cases of dimension when $n = 2, 3$. Furthermore, our setting allows for the treatment of all kinds of boundary conditions (i.e., periodic, no-slip, no-flux, etc) for (u, d) ; they will be incorporated in the weak formulation for the problem (1.1) and the information associated with the dissipation and smoothing operators from (1.2).

(ii) We establish general results on regularity, uniqueness and continuous dependence with respect to initial data for the family (1.1) in the general case when $\lambda_2 \neq 0$ and $\mu_1 \geq 0$.

(iii) We prove results on the existence of finite-dimensional of global attractors, and existence of exponential attractors (also known as inertial sets) for the entire three-parameter family (1.1) in the general case of (ii) and when $\theta > 0$. Due to loss of compactness of the semigroup associated with problem (1.1), the proofs require a completely different argument than the one given in [11], for a simplified Ericksen-Leslie family, and is based on a short trajectory type technique (see [10]).

(iv) We discuss the convergence, as time goes to infinity, of solutions of (1.1) to single equilibria. More precisely, by the Łojasiewicz–Simon technique we establish the convergence of any globally-defined weak solution of (1.1) with finite energy to a single steady state, regardless of whether uniqueness is known or not for (1.1), provided that the time-dependent body force g is asymptotically decaying in a precise way, i.e.,

$$\int_t^\infty \|g(s)\|_{H^{-\theta-\theta_2}}^2 ds \lesssim (1+t)^{-(1+\delta)}, \text{ for all } t \geq 0,$$

for some $\delta \in (0, 1)$. In particular, we show for any fixed initial datum $(u_0, d_0) \in H^{-\theta_2} \times H^1$, the corresponding trajectory $(u(t), d(t))$ satisfies

$$(1.6) \quad u(t) \rightarrow 0 \text{ weakly in } H^{-\theta_2} \text{ and } d(t) \rightarrow d_* \text{ strongly in } H^0,$$

as t tends to ∞ , where d_* is a steady-state of $A_1 d_* + f(d_*) = 0$. We emphasize that (1.6) holds for all weak solutions satisfying a suitable energy inequality, and so it holds in particular for the limit points of approximate solutions constructed within a numerical scheme. This result is also valid for the original (NS-EL) model (1.1) with $(\theta, \theta_1, \theta_2) = (1, 0, 0)$, $\chi = 0$ and extends a result obtained for a simplified version of the (NS-EL) model analyzed in [20]. Finally, we also give sufficient conditions for the model (1.1) in order to have a stronger convergence result in (1.6). More precisely, we show that

$$u(t) \rightarrow 0 \text{ strongly in } H^{-\theta_2} \text{ and } d(t) \rightarrow d_* \text{ strongly in } H^1,$$

provided that

$$\theta + \theta_2 \geq 1 \text{ and } d \text{ belongs to } L^\infty(\mathbb{R}_+; L^\infty(\Omega)).$$

(v) Exploiting the framework of [14] which is also extended in [11], the abstract mapping assumptions we employ for (1.1) are more general, and as a result do not require any specific

form of the parametrizations of A_0 , M , and Q , as in (1.2). As a consequence, our framework allows us to derive new results for a much larger three-parameter family of models that have not been explicitly studied elsewhere in detail. Finally, in Section 6 we give some conclusions about the abstract model and its connection to the standard models as introduced in Table 2.

The remainder of the paper is structured as follows. In Section 2, we establish our notation and give some basic preliminary results for the operators appearing in the general regularized model. In Section 3, we build some well-posedness results for the general model; in particular, we establish existence results (Section 3.1), regularity results (Section 3.2), and uniqueness and continuous dependence results (Section 3.3). In Section 4, we show existence of a finite-dimensional global attractor for the general model by employing the approach from [10]. In Section 5, we establish the eventual asymptotic stabilization as time goes to infinity of solutions to our regularized models, with the help from a Łojasiewicz–Simon technique. To make the paper sufficiently self-contained, our final Section 7 contains supporting material on Sobolev and Grönwall-type inequalities, and several other abstract results which are needed to prove our main results.

2. PRELIMINARY MATERIAL

2.1. The functional framework. We follow the same framework and notation as in [14] (cf. also [11]). To this end, let Ω be an n -dimensional smooth compact manifold with or without boundary and equipped with a volume form, and let $E \rightarrow \Omega$ be a vector bundle over Ω endowed with a Riemannian metric $h = (h_{ij})_{n \times n}$. With $C^\infty(E)$ denoting the space of smooth sections of E , let $\mathcal{V} \subseteq C^\infty(E)$ be a linear subspace, let $A_0 : \mathcal{V} \rightarrow \mathcal{V}$ be a linear operator, and let $B_0 : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ be a bilinear map. At this point \mathcal{V} is conceived to be an arbitrary linear subspace of $C^\infty(E)$; however, later on, we will impose some explicit restrictions on \mathcal{V} (see below). Furthermore, we let $\mathcal{W} \subseteq C^\infty(E)$ be a linear subspace and let $A_1 : \mathcal{W} \rightarrow \mathcal{W}$ be a linear operator satisfying various assumptions below. In order to define the variational setting for the phase-field component we also need to introduce the bilinear operators $R_0 : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{V}$, $B_1 : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{W}$, as follows:

$$(2.1) \quad B_1(u(x), d(x)) := Qu(x) \cdot \nabla d(x), \quad R_0(\psi(x), d(x)) := \psi(x) \cdot \nabla d(x).$$

Given the initial data $u_0 \in \mathcal{V}$, $d_0 \in \mathcal{W}$ and forcing term $g \in C^\infty(0, T; \mathcal{V})$ with $T > 0$, consider the following system

$$(2.2) \quad \begin{cases} \partial_t u + A_0 u + B_0(u, u) = R_0(A_1 d, d) + \operatorname{div}(\sigma_Q) + g, \\ \partial_t d + B_1(u, d) - \omega_Q d + \frac{\lambda_2}{\lambda_1} A_Q d = \frac{1}{\lambda_1} (A_1 d + \nabla_d W(d)), \\ u(0) = u_0, d(0) = d_0, \end{cases}$$

on the time interval $[0, T]$. Bearing in mind the model (1.1), we are mainly interested in bilinear maps of the form

$$(2.3) \quad B_0(v, w) = \bar{B}_0(Mv, Qw),$$

where M and Q are linear operators in \mathcal{V} that are relatively flexible, and \bar{B}_0 is a bilinear map fixing the underlying nonlinear structure of the fluid equation. In the following, denote $P : C^\infty(E) \rightarrow \mathcal{V}$ as the L^2 -orthogonal projector onto \mathcal{V} . When $\sigma_Q \equiv 0$, $\omega_Q \equiv 0$ and $\lambda_2 = 0$, the system (2.2) corresponds to a simplified (regularized) Ericksen–Leslie system that was fully investigated in [11].

We will study the regularized system (2.2) by extending it to function spaces that have weaker differentiability properties. To this end, we interpret (2.2) in a distributional sense, and need to continuously extend A_0 , A_1 and B_0, B_1 and R_0 to appropriate smoothness spaces. Namely, we employ the spaces $V^s = \operatorname{clos}_{H^s} \mathcal{V}$, $W^s = \operatorname{clos}_{H^s} \mathcal{W}$ (H^s denotes the Sobolev space of order s), which will informally be called Sobolev spaces in the sequel. The pair of spaces V^s and V^{-s} are equipped with the duality pairing $\langle \cdot, \cdot \rangle$, that is, the continuous extension of the L^2 -inner product to V^0 . Same applies to the triplet $W^s \subset W^0 = (W^0)^* \subset W^{-s}$. Moreover, we assume that there are self-adjoint *positive* operators Λ and A_1 , respectively, such that $\Lambda^s : V^s \rightarrow V^0$, $A_1^{s/2} : W^s \rightarrow W^0$ are isometries for any $s \in \mathbb{R}$, and Λ^{-1} , $(A_1)^{-1}$ are compact operators. For arbitrary real s , assume that A_0 , A_1 , M , and Q can be continuously extended so that

$$(2.4) \quad A_0 : V^s \rightarrow V^{s-2\theta}, \quad A_1 : W^s \rightarrow W^{s-2}, \quad M : V^s \rightarrow V^{s+2\theta_1}, \quad \text{and} \quad Q : V^s \rightarrow V^{s+2\theta_2},$$

are bounded operators. Again, we emphasize that the assumptions we will need for A_0 , M , and Q are more general, and do not require this particular form of the parametrization (see (2.5)–(2.7))

below). We will assume $\theta, \theta_2 \geq 0$ and no *a priori* sign restriction on θ_1 . The canonical norm in the Hilbert spaces V^s and W^s , respectively, will be denoted by the same quantity $\|\cdot\|_s$ whenever no further confusion arises, while we will use the notation $\|\cdot\|_{L^p}$ for the L^p -norm. Furthermore, we assume that A_0 and Q are both self-adjoint, and coercive in the sense that for $\beta \in \mathbb{R}$,

$$(2.5) \quad \langle A_0 w, \Lambda^{2\beta} w \rangle \geq c_{A_0} \|w\|_{\theta+\beta}^2 - C_{A_0} \|w\|_{\beta}^2, \quad w \in V^{\theta+\beta},$$

with $c_{A_0} = c_{A_0}(\beta) > 0$, and $C_{A_0} = C_{A_0}(\beta) \geq 0$, and that

$$(2.6) \quad \langle Qw, w \rangle \geq c_Q \|w\|_{-\theta_2}^2, \quad w \in V^{-\theta_2},$$

with $c_Q > 0$. We also assume that

$$(2.7) \quad \langle A_0 w, Qw \rangle \geq c_{A_0} \|w\|_{\theta-\theta_2}^2, \quad w \in V^{\theta-\theta_2},$$

Note that if $\theta = 0$, (2.5) is strictly speaking not coercivity and follows from the boundedness of A_0 , and note also that (2.6) implies the invertibility of Q .

One may typically consider the following examples of operators occurring in various combinations in (2.2).

Example 2.1. (a) When Ω is a closed Riemannian manifold, and $E = T\Omega$ the tangent bundle, an example of \mathcal{V} is $\mathcal{V}_{\text{per}} \subseteq \{u \in C^\infty(T\Omega) : \operatorname{div}(u) = 0\}$, a subspace of the divergence-free functions. The space of periodic functions with vanishing mean on a torus \mathbb{T}^n is a special case of this example. In this case, one typically has $A_0 = (-\Delta)^\theta$, $M = (I - \alpha^2 \Delta)^{-\theta_1}$, $Q = (I - \alpha^2 \Delta)^{-\theta_2}$ and $A_1 = -\Delta$, as operators that satisfy (2.4), cf. [14, Example 2.1, (a)].

(b) When Ω is a compact Riemannian manifold with boundary Γ and again $E = T\Omega$ the tangent bundle, a typical example of \mathcal{V} is $\mathcal{V}_{\text{hom}} = \{u \in C_0^\infty(T\Omega) : \operatorname{div}(u) = 0\}$. In this case, one may consider the choices $A_0 = (-P\Delta)^\theta$, $A_1 = -\Delta$, $M = (I - \alpha^2 P\Delta)^{-\theta_1}$, and $Q = (I - \alpha^2 P\Delta)^{-\theta_2}$, respectively, as operators satisfying (2.4), cf. [14, Example 2.1, (b)].

(c) Let Ω be connected Riemannian n -dimensional manifold with non-empty (sufficiently smooth) boundary $\partial\Omega$. Define $A_1 = -\Delta$, as the Laplacian of the metric h , acting on

$$D(A_1) = \{\phi \in W^2 : \phi = 0 \text{ on } \partial\Omega\},$$

where in local coordinates $\{x_i\}_{i=1}^n$, the Laplacian reads

$$\Delta(\cdot) = \frac{1}{\sqrt{\det(h)}} \sum_{i,j=1}^n \partial_{x_j} \left(h^{ij} \sqrt{\det(h)} \partial_{x_i}(\cdot) \right);$$

the matrix (h^{ij}) denotes the inverse of h . We have that A_1 is a positive self-adjoint operator on W^0 .

Example 2.2. In Example 2.1 above, the bilinear map \bar{B}_0 can be taken to be

$$(2.8) \quad \bar{B}_{0\chi}(v, w) = P[(v \cdot \nabla)w + \chi(\nabla v^T)v],$$

which correspond to the models with $\chi \in \{0, 1\}$ as introduced in the system (1.1).

To refer to the above examples, let us further introduce the shorthand notation:

$$(2.9) \quad B_{0\chi}(v, w) = \bar{B}_{0\chi}(Mv, Qw), \quad \chi \in \{0, 1\}.$$

For clarity, we list in Table 1 the corresponding values of the parameters and bilinear maps discussed above for special cases as given by (1.2).

Next, we denote the trilinear forms

$$(2.10) \quad b_0(u, v, w) = \langle B_0(u, v), w \rangle, \quad b_1(u, d, \psi) = \langle B_1(u, d), \psi \rangle,$$

and similarly the forms $\bar{b}_{0\chi}$ and $b_{0\chi}$, following (2.1), (2.8) and (2.9). Then our notion of *weak solution* for problem (2.2) can be formulated as follows.

Definition 2.3. Let $g(t) \in L^2(0, T; V^{-s})$, for some $s \in \mathbb{R}$ and

$$(u_0, d_0) \in \mathcal{Y}_{\theta_2} := \begin{cases} V^{-\theta_2} \times W^1, & \text{if } \lambda_2 \neq 0, \\ V^{-\theta_2} \times (W^1 \cap \{d_0 \in L^\infty(\Omega) : \|d_0\|_{L^\infty} \leq 1\}), & \text{if } \lambda_2 = 0. \end{cases}$$

Find a pair of functions

$$(2.11) \quad (u, d) \in L^\infty(0, T; \mathcal{Y}_{\theta_2}) \cap L^2(0, T; V^{\theta-\theta_2} \times W^2)$$

TABLE 1. Values of the parameters θ , θ_1 and θ_2 , and the particular form of the bilinear map B_0 for some special cases of the model (2.2). (The bilinear maps B_{00} and B_{01} are as in (2.9)). More precisely, it allows us to include the special cases when u satisfies the Navier-Stokes equation (NSE), the Leray- α -equation, the modified Leray- α -equation (ML), the simplified Bardina model (SBM), the Navier-Stokes-Voigt equation (NSV) and the Lagrangian averaged Navier-Stokes- α model (NS- α -model).

Model	NSE-EL	Leray-EL- α	ML-EL- α	SBM-EL	NSV-EL	NS-EL- α
θ	1	1	1	1	0	1
θ_1	0	1	0	1	1	0
θ_2	0	0	1	1	1	1
B_0	B_{00}	B_{00}	B_{00}	B_{00}	B_{00}	B_{01}

such that

$$(2.12) \quad \partial_t u \in L^p(0, T; V^{-\gamma}), \quad \partial_t d \in L^2(0, T; W^{-2})$$

for some $p > 1$ and $\gamma \geq 0$, such that (u, d) fulfills $u(0) = u_0$, $d(0) = d_0$ and satisfies

$$(2.13) \quad \begin{aligned} & \int_0^T \left(-\langle u(t), w'(t) \rangle + \langle A_0 u(t), w(t) \rangle + b_0(u(t), u(t), w(t)) \right) dt \\ &= \int_0^T \left(\langle g(t), w(t) \rangle + \langle R_0(A_1 d(t), d(t)), w(t) \rangle - \langle \sigma_Q, \nabla w(t) \rangle \right) dt, \end{aligned}$$

$$(2.14) \quad \begin{aligned} & \int_0^T \left(-\langle d(t), \psi'(t) \rangle + \langle \mu(t), \psi(t) \rangle + b_1(u(t), d(t), \psi(t)) \right) dt \\ &= \int_0^T \left(\langle \omega_Q d, \psi(t) \rangle - \frac{\lambda_2}{\lambda_1} \langle A_Q d, \psi(t) \rangle \right) dt, \end{aligned}$$

for any $(w, \psi) \in C_0^\infty(0, T; \mathcal{V} \times \mathcal{W})$, with $\mu(t) := -\lambda_1^{-1}(A_1 d(t) + \nabla_d W(d(t)))$ a.e. on $\Omega \times (0, T)$.

Remark 2.1. As far as the interpretation of the initial conditions $u(0) = u_0$, $d(0) = d_0$ is concerned, note that properties (2.11)-(2.12) imply that $u \in C(0, T; V^{-\gamma})$ and $d \in C(0, T; W^0)$. Thus, the initial conditions are satisfied in a weak sense. All kinds of boundary conditions (i.e., periodic, no-slip, no-flux, etc) for (u, d) can be treated and will be included in our analysis; they will be incorporated in the weak formulation for the problem (2.2). On the other hand, for those values of $(\theta, \theta_1, \theta_2)$ from Table 1 we recover some specific regularization models given by (1.1) for the particular choices of the operators A_0, M, Q and χ in (1.2), as listed in Table 1.

TABLE 2. Some special cases of the model (1.1) with $\alpha > 0$, and with $\Pi = (I - \alpha^2 \Delta)^{-1}$.

Model	NSE-EL	Leray-EL- α	ML-EL- α	SBM-EL	NSV-EL	NS-EL- α
A_0	$-\mu_4 \Delta$	$-\mu_4 \Delta$	$-\mu_4 \Delta$	$-\mu_4 \Delta$	$-\mu_4 \Delta \Pi$	$-\mu_4 \Delta$
M	I	Π	I	Π	Π	Π
Q	I	I	Π	Π	Π	I
χ	0	0	0	0	0	1

Throughout the paper, $C \geq 0$ will denote a *generic* constant whose further dependence on certain quantities will be specified on occurrence. The value of the constant can change even within the same line. Furthermore, we introduce the notation $a \lesssim b$ to mean that there exists a constant $C > 0$ such that $a \leq Cb$. This notation will be used when the explicit value of C is irrelevant or tedious to write down.

2.2. Energy estimates and solutions. The system (2.2) admits a *total regularized energy*, consisting of kinetic and potential energies, given by

$$(2.15) \quad E_Q(t) = \frac{1}{2} \left[\langle u(t), Qu(t) \rangle + \|A_1^{1/2} d(t)\|_{L^2}^2 \right] + \int_{\Omega} W(d) dx.$$

In particular, for the smoothed systems introduced in (1.1), the total energy E_Q can be identified with the energy of the original NSE-EL system under suitable boundary conditions. Furthermore, in the case of the α -models from Table 1, the invariant E_Q reduces, as $\alpha \rightarrow 0$, to the dissipated energy E_I of the NSE-EL system. In order to show that E_Q is an ideal invariant for the system (2.2), we need to perform some basic energy estimates and computations. In what follows, we will always force the following

Assumption on \mathcal{V} : For a given smooth tensor $\Xi = \Xi(x) \in \mathbb{R}^{n \times n}$, we require that the following identity holds:

$$(2.16) \quad \langle \operatorname{div}(\Xi), v \rangle + \langle \Xi, \nabla v \rangle = 0,$$

for any $v = Qu \in \mathcal{V}$. In particular, such an assumption always holds provided that $\mathcal{V} = \mathcal{V}_{per}$ is the space of periodic (divergence-free) functions with vanishing mean on a torus $\Omega = \mathbb{T}^n$, see Example 2.1, (a). Clearly, (2.16) will also hold in function spaces $V^s = \operatorname{clos}_{H^s} \mathcal{V}$, $s \geq 1$, that have weaker differentiability properties. For more details on the nature of this assumption, we refer the reader to Section 6.

Energy estimates: In order to deduce a particular energy identity, we will also assume that $b_0(u, u, Qu) = 0$, for any $Qu \in \mathcal{V}$; pairing the first equation of (2.2) with Qu and the second equation with $A_1 d + \nabla_d W(d)$, respectively, by virtue of (2.16) we deduce

$$(2.17) \quad \begin{aligned} & \frac{d}{dt} E_Q(t) + \langle A_0 u, Qu \rangle - \frac{1}{\lambda_1} \|A_1 d + \nabla_d W(d)\|_{L^2}^2 \\ &= -\langle \sigma_Q, \nabla(Qu) \rangle + \langle \omega_Q d, A_1 d + \nabla_d W(d) \rangle - \frac{\lambda_2}{\lambda_1} \langle A_Q d, A_1 d + \nabla_d W(d) \rangle \\ &+ \langle g(t), Qu \rangle, \end{aligned}$$

for $t \in (0, T)$, for any fixed but otherwise arbitrary $T > 0$. In order to simplify this identity further, we assume as in [22] that the coefficients $\lambda_1, \lambda_2, \mu_1, \dots, \mu_6$ obey certain constraints:

$$(2.18) \quad \lambda_1 < 0,$$

$$(2.19) \quad \mu_1 \geq 0, \quad \mu_4 > 0,$$

$$(2.20) \quad \mu_5 + \mu_6 \geq 0,$$

$$(2.21) \quad \lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6.$$

Then, we insert the expression for the Leslie stress tensor σ_Q from (1.5) and perform analogous computations as in [16, 22], relying on the symmetric properties of A_Q and anti-symmetric properties of ω_Q . We obtain after some lengthy but standard transformations that

$$(2.22) \quad \begin{aligned} & \frac{d}{dt} E_Q(t) + \langle A_0 u, Qu \rangle - \frac{1}{\lambda_1} \|A_1 d + \nabla_d W(d)\|_{L^2}^2 + \mu_1 \|d^T A_Q d\|_{L^2}^2 \\ &= -(\mu_2 + \mu_3) \langle d \otimes \mathcal{N}_Q, A_Q \rangle - (\mu_5 + \mu_6) \|A_Q d\|_{L^2}^2 \\ &- \frac{\lambda_2}{\lambda_1} \langle A_Q d, \lambda_1 \mathcal{N}_Q + \lambda_2 A_Q d \rangle + \langle g(t), Qu \rangle. \end{aligned}$$

We recall that (2.18)–(2.21) are always necessary in order for the energy E_Q , $Q = I$, of the system (1.1) to be nonincreasing in the absence of external forces (cf. [22]). In addition, we'll also assume two different sets of hypotheses on the coefficients according to [22].

- **Case 1** (with Parodi's relation). Suppose that (2.18)–(2.21) are satisfied. Moreover, we enforce the following Parodi's relation $\mu_2 + \mu_3 = \mu_6 - \mu_5$ and

$$(2.23) \quad \frac{(\lambda_2)^2}{-\lambda_1} \leq \mu_5 + \mu_6.$$

- **Case 2** (without Parodi's relation). Suppose that (2.18)–(2.21) are satisfied. Moreover, we assume

$$(2.24) \quad |\lambda_2 - \mu_2 - \mu_3| < 2\sqrt{-\lambda_1} \sqrt{\mu_5 + \mu_6}.$$

In **Case 1**, it turns out that the regularized energy E_Q satisfies for smooth solutions the identity

$$(2.25) \quad \frac{d}{dt} E_Q(t) + \langle A_0 u, Qu \rangle - \frac{1}{\lambda_1} \|A_1 d + \nabla_d W(d)\|_{L^2}^2 + \mu_1 \|d^T A_Q d\|_{L^2}^2$$

$$= - \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) \|A_Q d\|_{L^2}^2 + \langle g(t), Qu \rangle,$$

while in the **Case 2**, it obeys

$$(2.26) \quad \begin{aligned} & \frac{d}{dt} E_Q(t) + \langle A_0 u, Qu \rangle + \mu_1 \|d^T A_Q d\|_{L^2}^2 - \frac{3\lambda_1}{4} \|\mathcal{N}_Q\|_{L^2}^2 \\ & \leq - \left(\mu_5 + \mu_6 + \frac{(\lambda_2 - (\mu_2 + \mu_3))^2}{\lambda_1} \right) \|A_Q d\|_{L^2}^2 + \langle g(t), Qu \rangle. \end{aligned}$$

Next, for every $\varepsilon > 0$ we have

$$\langle g(t), Qu \rangle \leq \varepsilon^{-1} \|g(t)\|_{\theta-\theta_2}^2 + \varepsilon \|Q\|_{-\theta_2; \theta_2}^2 \|u\|_{\theta-\theta_2}^2.$$

Employing now the condition (2.7) for the operator A_0 , we can absorb this term on the right-hand side of (2.25)-(2.26). In either case, by integrating the resulting relation on the time interval (s, t) , we easily derive that E_Q satisfies an energy inequality. More precisely, owing to (2.7) there holds a.e. $t > 0$,

$$(2.27) \quad \begin{aligned} & \frac{d}{dt} E_Q(t) + \frac{c_{A_0}}{2} \|u(t)\|_{\theta-\theta_2}^2 - \frac{1}{\lambda_1} \|A_1 d(t) + \nabla_d W(d(t))\|_{L^2}^2 + \mu_1 \|(d^T A_Q d)(t)\|_{L^2}^2 \\ & \leq \frac{2\|Q\|_{-\theta_2; \theta_2}^2}{c_{A_0}} \|g(t)\|_{\theta-\theta_2}^2. \end{aligned}$$

It follows from (2.27) by integration over $(0, t)$ that $(u(t), d(t))$ belongs to the functional class (2.11) given $g \in L^2(0, T; V^{-\theta-\theta_2})$. We note that $E_Q(0) < \infty$ is equivalent to having $(u_0, d_0) \in \mathcal{Y}_{\theta_2}$.

We now introduce another notion of weak solutions which is also essential in our subsequent study.

Definition 2.4. *Let $\lambda_1, \lambda_2, \mu_1, \dots, \mu_6$ satisfy the above assumptions according to the **Cases 1-2**. By an energy solution we will mean a weak distributional solution (u, d) , satisfying the weak formulation (2.13)-(2.14) and obeying the energy inequality (2.27) according to the Cases 1 and 2, respectively.*

It is worth pointing out that, by virtue of (2.25)-(2.26), energy solutions of the regularized Ericksen-Leslie system (2.2) satisfy:

$$(2.28) \quad \begin{cases} A_1 d + \nabla_d W(d) \in L^2(0, T; L^2(\Omega)), & A_Q d \in L^2(0, T; L^2(\Omega)), \\ \mathcal{N}_Q \in L^2(0, T; L^2(\Omega)), & d^T A_Q d \in L^2(0, T; L^2(\Omega)). \end{cases}$$

Indeed, the energy dissipations provided by the inequalities in (2.25) and (2.26) are equivalent because of definitions (1.3), (1.4) and the equation for the director field d from (2.2) (see also [22]). Such knowledge will also become important in the study of global regularity.

3. WELL-POSEDNESS RESULTS

Analogous to the theory of regularized flows we have developed for a simplified Ericksen-Leslie model in [11], we begin to devise a solution theory for the general three-parameter family of regularized models from (1.1). We begin to establish existence and regularity results, and under appropriate assumptions uniqueness and stability. At the end of the proof of each theorem, we give the corresponding conditions for $(\theta, \theta_1, \theta_2)$ which allow us not only to establish old results but also new results in the literature, especially for the cases listed in Table 1. The analysis in this section is divided mainly into two parts according to whether $\lambda_2 = 0$ or $\lambda_2 \neq 0$.

3.1. Existence of weak solutions. In this subsection, we establish sufficient conditions for the existence of energy solutions to the problem (2.2) (cf. Definition 2.4). As noted previously, in the case when $\lambda_2 = 0$, a maximum principle holds for the director field d of any weak solution.

Proposition 3.1. *Suppose that $b_1(v, \psi, \psi) = 0$, for any $v \in V^{\theta-\theta_2}$, $\psi \in W^1$. Let $d_0 \in L^\infty(\Omega)$ such that $\|d_0\|_{L^\infty(\Omega)} \leq 1$. Then, for any weak solution (u, d) to problem (2.2) in the sense of Definition 2.3, we have $d \in L^\infty(0, T; L^\infty(\Omega))$ and*

$$(3.1) \quad |d(x, t)| \leq \|d_0\|_{L^\infty(\Omega)}, \text{ a.e. on } \Omega \times (0, T).$$

Proof. The inequality in (3.1) follows from a straightforward application of the weak maximum principle, since $\lambda_2 = 0$ and the tensor ω_Q is skew-symmetric. A Moser type of iteration argument also gives the desired regularity $d \in L^\infty(0, T; L^\infty(\Omega))$ (see, e.g., [4, Lemma 9.3.1]). \square

It is worth emphasizing that when $\lambda_2 \neq 0$, the inequality (3.1) is generally not expected to hold.

Theorem 3.2. *Assume Proposition 3.1 only when $\lambda_2 = 0$ and let the following conditions hold.*

- i) $(u_0, d_0) \in \mathcal{Y}_{\theta_2}$ with any $\theta_2 \geq 0$ and $g \in L^2(0, T; V^{-\theta-\theta_2})$, $T > 0$.
- ii) $b_0(v, v, Qv) = 0$, for any $v \in V^{\theta-\theta_2}$;
- iii) $b_0 : V^{\bar{\sigma}_1} \times V^{\bar{\sigma}_2} \times V^{\bar{\gamma}} \rightarrow \mathbb{R}$ is bounded for some $\bar{\sigma}_i < \theta - \theta_2$, $i = 1, 2$, and $\bar{\gamma} \geq \gamma$;
- iv) $b_0 : V^{\sigma_1} \times V^{\sigma_2} \times V^\gamma \rightarrow \mathbb{R}$ is bounded for some $\sigma_i \in [-\theta_2, \theta - \theta_2]$, $i = 1, 2$, and $\gamma \in [\theta + \theta_2, \infty) \cap (\theta_2, \infty) \cap (1 + n/6, \infty)$;

Then, there exists at least one energy solution (u, d) satisfying (2.11)-(2.12), (2.28) such that

$$(3.2) \quad p = \begin{cases} \min\{2, \frac{2\theta}{\sigma_1 + \sigma_2 + 2\theta_2}, \frac{4(6-n)}{12-n}\}, & \text{if } \theta > 0, \\ \frac{4(6-n)}{12-n}, & \text{if } \theta = 0. \end{cases}$$

Proof. We rely on a Galerkin approximation scheme by borrowing ideas from [14]. To this end, let $\{V_m : m \in \mathbb{N}\} \subset V^{\theta-\theta_2}$, $\{W_m : m \in \mathbb{N}\} \subset D(A_1) \cap L^\infty(\Omega)$ be sequences of finite dimensional (smooth) subspaces of $V^{\theta-\theta_2}$ and $D(A_1)$, respectively, such that

- (1) $V_m \subset V_{m+1}$, $W_m \subset W_{m+1}$, for all $m \in \mathbb{N}$;
- (2) $\cup_{m \in \mathbb{N}} V_m$ is dense in $V^{\theta-\theta_2}$, and $\cup_{m \in \mathbb{N}} W_m$ is dense in $D(A_1)$;
- (3) For $m \in \mathbb{N}$, with $\tilde{V}_m = QV_m \subset V^{\theta+\theta_2}$, the projectors $P_m : V^{\theta-\theta_2} \rightarrow V_m$, $S_m : D(A_1) \rightarrow W_m$, defined by

$$\begin{aligned} \langle P_m v, w_m \rangle &= \langle v, w_m \rangle, & w_m \in \tilde{V}_m, v \in V^{\theta-\theta_2}, \\ \langle S_m d, \psi_m \rangle &= \langle d, \psi_m \rangle, & \psi_m \in W_m, d \in D(A_1), \end{aligned}$$

are uniformly bounded as maps from $V^{-\gamma} \rightarrow V^{-\gamma}$ and $W^{-2} \rightarrow W^{-2}$, respectively.

Such sequences can be constructed e.g., by using the eigenfunctions of the isometries $\Lambda^{1+\theta} : V^{1+\theta-\theta_2} \rightarrow V^{-\theta_2}$, $A_1 : D(A_1) \rightarrow W^0$. Consider the problem of finding $(u_m, d_m) \in C^1(0, T; V_m \times W_m)$ such that for all $(w_m, \psi_m) \in \tilde{V}_m \times W_m$,

$$(3.3) \quad \begin{cases} \langle \partial_t u_m, w_m \rangle + \langle A_0 u_m, w_m \rangle + b_0(u_m, u_m, w_m) \\ = \langle g, w_m \rangle + \langle R_0(A_1 d_m, d_m) + \operatorname{div}(\sigma_Q^m), w_m \rangle, \\ \langle \partial_t d_m, \psi_m \rangle + b_1(u_m, d_m, \psi_m) + \langle \omega_Q^m d_m + \frac{\lambda_2}{\lambda_1} A_Q^m d_m, \psi_m \rangle \\ = \frac{1}{\lambda_1} \langle A_1 d_m + \nabla_d W(d_m), \psi_m \rangle, \\ \langle u_m(0), w_m \rangle = \langle u_0, w_m \rangle, \\ \langle d_m(0), \psi_m \rangle = \langle d_0, \psi_m \rangle, \end{cases}$$

where

$$\begin{aligned} A_Q^m &= \frac{1}{2}(\nabla(Qu_m) + \nabla^T(Qu_m)), \quad \omega_Q^m = \frac{1}{2}(\nabla(Qu_m) - \nabla^T(Qu_m)), \\ \dot{d}_m &= \partial_t d_m + Qu_m \cdot \nabla d_m, \quad \mathcal{N}_Q^m = \dot{d}_m - \omega_Q^m d_m \end{aligned}$$

and

$$\sigma_Q^m = \mu_1(d_m^T A_Q^m d_m) d_m \otimes d_m + \mu_2 \mathcal{N}_Q^m \otimes d_m + \mu_3 d_m \otimes \mathcal{N}_Q^m + \mu_5(A_Q^m d_m) \otimes d_m + \mu_6 d_m \otimes (A_Q^m d_m).$$

Choosing a basis for $V_m \times W_m$, one sees that the system (3.3) is an initial value problem for a system of ODE's. By definition, the operator Q is invertible so that the standard ODE theory gives a unique solution to (3.3), which is locally-defined in time. Using the definition of Q once more, one checks that

$$c_Q \|u_m(0)\|_{-\theta_2}^2 \leq \langle u_m(0), Qu_m(0) \rangle = \langle u(0), Qu_m(0) \rangle \leq \|u(0)\|_{-\theta_2} \|Qu_m(0)\|_{\theta_2},$$

so that $\|u_m(0)\|_{-\theta_2}$ is uniformly bounded. Similarly, one shows that $\|d_m(0)\|_1$ is uniformly bounded.

Now in the first and second equalities of (3.3), taking $w_m = Qu_m$ and $\psi_m = A_1 d_m + \nabla_d W(d_m)$, respectively, and using the a priori estimates established earlier in (2.25)-(2.28), one derives that

the solution (u_m, d_m) is uniformly bounded in $L^\infty(0, T; \mathcal{Y}_{\theta_2}) \cap L^2(0, T; V^{\theta-\theta_2} \times D(A_1))$. Moreover, the terms $A_1 d_m + \nabla_d W(d_m)$, $A_Q^m d_m$, \mathcal{N}_Q^m and $d_m^T A_Q^m d_m$ are also uniformly bounded in $L^2(0, T; L^2(\Omega))$. Thus, passing to a subsequence, one has

$$(3.4) \quad \begin{cases} (u_m, d_m) \rightarrow (u, d) \text{ weak-star in } L^\infty(0, T; \mathcal{Y}_{\theta_2}), \\ (u_m, d_m) \rightarrow (u, d) \text{ weakly in } L^2(0, T; V^{\theta-\theta_2} \times D(A_1)). \end{cases}$$

Passing to the limit as $m \rightarrow \infty$ in (3.3) requires the use of compactness arguments. To this end, we start by estimating $\|\partial_t u_m\|_{-\gamma}$ and $\|\partial_t d_m\|_{-2}$, respectively. The first equation in (3.3) may be recast as:

$$(3.5) \quad \partial_t u_m + P_m A_0 u_m + P_m B_0(u_m, u_m) = P_m (g + R_0(A_1 d_m, d_m) + \operatorname{div}(\sigma_Q^m)).$$

Consequently,

$$(3.6) \quad \begin{aligned} \|\partial_t u_m\|_{-\gamma} &\leq \|P_m A_0 u_m\|_{-\gamma} + \|P_m B_0(u_m, u_m)\|_{-\gamma} + \|P_m g\|_{-\gamma} \\ &\quad + \|P_m R_0(A_1 d_m, d_m)\|_{-\gamma} + \|P_m \operatorname{div}(\sigma_Q^m)\|_{-\gamma} \\ &\lesssim \|u_m\|_{\theta-\theta_2} + \|P_m B_0(u_m, u_m)\|_{-\gamma} + \|g\|_{-\theta-\theta_2} \\ &\quad + \|P_m R_0(A_1 d_m, d_m)\|_{-\gamma} + \|P_m \operatorname{div}(\sigma_Q^m)\|_{-\gamma}. \end{aligned}$$

Now, thanks to the boundedness of B_0 (see (iv)), it follows as in [14, Theorem 3.1] that

$$(3.7) \quad \|P_m B_0(u_m, u_m)\|_{-\gamma} \lesssim \|u_m\|_{\sigma_1} \|u_m\|_{\sigma_2}.$$

If $\theta = 0$, then the norms in the right hand side are the $V^{-\theta_2}$ -norm which is uniformly bounded. On the other hand, if $\theta > 0$, then by interpolation, one gets

$$(3.8) \quad \|u_m\|_{\sigma_i} \lesssim \|u_m\|_{-\theta_2}^{1-\lambda_i} \|u_m\|_{\theta-\theta_2}^{\lambda_i}, \quad \lambda_i = \frac{\sigma_i + \theta_2}{\theta}, \quad i = 1, 2,$$

so that

$$(3.9) \quad \|P_m B_0(u_m, u_m)\|_{-\gamma} \lesssim \|u_m\|_{-\theta_2}^{2-\lambda_1-\lambda_2} \|u_m\|_{\theta-\theta_2}^{\lambda_1+\lambda_2} \lesssim \|u_m\|_{\theta-\theta_2}^{\lambda_1+\lambda_2}.$$

Hence, with $\lambda := \lambda_1 + \lambda_2 = \frac{\sigma_1 + \sigma_2 + 2\theta_2}{\theta}$ if $\theta > 0$, and with $\lambda = 1$ if $\theta = 0$, we get

$$(3.10) \quad \int_0^T \|P_m B_0(u_m, u_m)\|_{-\gamma}^p dt \lesssim \|u_m\|_{L^p(V^{\theta-\theta_2})}^p + \|u_m\|_{L^{p\lambda}(V^{\theta-\theta_2})}^p.$$

The first term on the right-hand side is bounded uniformly when $p \leq 2$. The second term is bounded if $p\lambda \leq 2$, that is $p \leq 2/\lambda$. We conclude that $P_m B_0(u_m, u_m)$ is uniformly bounded in $L^p(V^{-\gamma})$, with $p = \min\{2, 2/\lambda\}$. Concerning a uniform bound for $P_m R_0(A_1 d_m, d_m)$ in $L^2(V^{-\gamma})$, we argue as in [11, Theorem 3.2] to derive that

$$(3.11) \quad \|P_m R_0(A_1 d_m, d_m)\|_{-\gamma}^2 \lesssim \|A_1 d_m\|_{L^2}^2 \|d_m\|_1^2$$

provided that $\gamma > 1 + \frac{n}{6} \geq \frac{n}{2}$. As for the remaining term in (3.6), one has:

$$(3.12) \quad \begin{aligned} \|P_m \operatorname{div}(\sigma_Q^m)\|_{-\gamma} &\lesssim \mu_1 \|\operatorname{div}(d_m^T A_Q^m d_m) d_m \otimes d_m)\|_{-\gamma} \\ &\quad + \|\operatorname{div}(\mu_2 \mathcal{N}_Q^m \otimes d_m + \mu_3 d_m \otimes \mathcal{N}_Q^m)\|_{-\gamma} \\ &\quad + \|\operatorname{div}(\mu_5 (A_Q^m d_m) \otimes d_m + \mu_6 d_m \otimes (A_Q^m d_m))\|_{-\gamma} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We'll just estimate I_1 ; estimating I_2 and I_3 follows suit. To this end, let $\varphi_m \in V^\gamma \subset W^{1,3}$ with $\|\varphi_m\|_\gamma = 1$ and $\gamma > \frac{n}{6} + 1$. Then

$$(3.13) \quad \begin{aligned} |\langle \operatorname{div}(d_m^T A_Q^m d_m) d_m \otimes d_m, \varphi_m \rangle| &= |\langle (d_m^T A_Q^m d_m) d_m \otimes d_m, \nabla \varphi_m \rangle| \\ &\lesssim \|d_m^T A_Q^m d_m\|_{L^2} \|A_1^{1/2} d_m\|_{L^2}^{2(1-\delta_n)} \|A_1 d_m\|_{L^2}^{2\delta_n} \end{aligned}$$

with $\delta_n = \frac{n}{4(6-n)}$. Similarly, we have

$$(3.14) \quad I_2 \lesssim \|\mathcal{N}_Q^m\|_{L^2} \|d_m\|_1 \text{ and } I_3 \lesssim \|A_Q^m d_m\|_{L^2} \|d_m\|_1.$$

Consequently, on account of the estimates (3.10)-(3.14) and thanks to Hölder's inequality, it follows from (3.6) that $\partial_t u_m$ is uniformly bounded in $L^p(0, T; V^{-\gamma})$, provided that $p \leq 2$, $p \leq \frac{4(6-n)}{12-n}$ and

$p \leq 2\lambda$ if $\theta > 0$, and $p \leq \frac{4(6-n)}{12-n}$ for $\theta = 0$. To estimate $\partial_t \phi_m$, we recast the second equation in (3.3) as

$$\begin{aligned} & \partial_t d_m + S_m B_1(u_m, d_m) + S_m \omega_Q^m d_m + \frac{\lambda_2}{\lambda_1} S_m A_Q^m d_m \\ &= \frac{1}{\lambda_1} S_m A_1 d_m + S_m \nabla_d W(d_m). \end{aligned}$$

It follows from the uniform boundedness of S_m that

$$(3.15) \quad \begin{aligned} \|\partial_t d_m\|_{-2} &\lesssim \|B_1(u_m, d_m)\|_{-2} + \|\omega_Q^m d_m\|_{-2} + \|A_Q^m d_m\|_{-2} \\ &\quad + \|A_1 d_m + \nabla_d W(d_m)\|_{-2}. \end{aligned}$$

Thanks to the Hahn-Banach theorem, Hölder's inequality and a proper Sobolev embedding theorem, we can argue as in [11, Theorem 3.2] to get the following estimates:

$$(3.16) \quad \begin{aligned} \|B_1(u_m, d_m)\|_{-2} &= \langle Qu_m \cdot \nabla d_m, \varphi_m \rangle, \varphi_m \in V^2, \|\varphi_m\|_2 = 1 \\ &\lesssim \|Qu_m\|_{\theta_2} \|\nabla d_m\|_1 \|\varphi_m\|_2 \\ &\lesssim \|u_m\|_{-\theta_2} \|A_1 d_m\|_{L^2}, \text{ since } \theta_2 \geq 0, \end{aligned}$$

and, using Einstein's summation convention,

$$(3.17) \quad \begin{aligned} \|\omega_Q^m d_m\|_{-2} &= \langle \omega_Q^m d_m, \psi_m \rangle, \psi_m \in V^2, \|\psi_m\|_2 = 1 \\ &= \left\langle (\omega_Q^m)_{ij} d_{mj}, \psi_{mi} \right\rangle = \left\langle (\omega_Q^m)_{ij}, d_{mj} \psi_{mi} \right\rangle \\ &\lesssim \|(\omega_Q^m)_{ij}\|_{-1} \|d_{mj} \psi_{mi}\|_1 \\ &\lesssim \|(\omega_Q^m)_{ij}\|_{-1} \|d_m\|_2. \end{aligned}$$

Now, since $\theta_2 \geq 0$, it holds

$$\|(\omega_Q^m)_{ij}\|_{-1} \lesssim \|(\nabla Q u_m)_{ij}\|_{-1} \lesssim \|Q u_m\|_0 \lesssim \|u_m\|_{-\theta_2}.$$

Substituting this bound into (3.17), we easily arrive at the bound

$$(3.18) \quad \|\omega_Q^m d_m\|_{-2} \lesssim \|u_m\|_{-\theta_2} \|A_1 d_m\|_{L^2},$$

It follows from (3.15)-(3.18) and earlier estimates that $\partial_t d_m$ is uniformly bounded in $L^2(0, T; V^{-2})$. With the estimates for $\partial_t u_m$ and $\partial_t d_m$, we now have the required ingredients for the application of the Aubin-Lions-Simon compactness theorem (see, e.g., [14, Appendix]). In particular, we can infer the existence of a limit couple

$$(u, d) \in C(0, T; V^{-\gamma} \times W^0) \cap L^\infty(0, T; \mathcal{Y}_{\theta_2})$$

such that, in addition to (3.4), we also have

$$(3.19) \quad \begin{cases} (u_m, d_m) \rightarrow (u, d) \text{ strongly in } L^2(0, T; V^s \times W^{2-}) \\ d_m \rightarrow d \text{ strongly in } C(0, T; W^{1-}), \end{cases}$$

for any $s < \theta - \theta_2$, where W^{s-} denotes $W^{s-\delta}$, for some sufficiently small $\delta \in (0, s]$.

We are now able to pass to the limit in all the nonlinear terms of (3.3) so that this limit couple (u, d) indeed satisfies the weak formulation (2.13)-(2.14) of Definition 2.3. This is standard procedure and so we leave the details to the interested reader. However, we refer the reader to [14, Theorem 3.1] for passage to the limit in the equation for the velocity and to [11, Theorem 3.2] for passage to the limit in the elastic (Ericksen) stress tensor R_0 . The proof of the theorem is now finished. \square

Our theorem covers the following special cases listed in Table 1.

Remark 3.1. Let $\theta + \theta_1 > \frac{1}{2}$ and recall that $\theta, \theta_2 \geq 0$. By [14, Proposition 2.5], the trilinear form b_{00} , defined by (2.9)-(2.10), fulfills the hypotheses (ii)-(iv) of Theorem 3.2 for $-\gamma \leq \theta - \theta_2 - 1$ with $-\gamma < \min\{2\theta + 2\theta_1 - \frac{n+2}{2}, \theta - \theta_2 + 2\theta_1, \theta + \theta_2 - 1\}$. Similarly, the trilinear form b_{01} satisfies (ii)-(iv) for $-\gamma \leq \theta - \theta_2 - 1$ with $-\gamma < \min\{2\theta + 2\theta_1 - \frac{n+2}{2}, \theta - \theta_2 + 2\theta_1 - 1, \theta + \theta_2\}$. In particular, our result yields the global existence of a weak energy solution for both the inviscid and viscous Leray-EL- α models in three space dimensions, and for all the other regularized models listed in

Table 1. As far as we know, except for the 3D NSE-EL system reported in [8], none of these results have been reported previously.

Remark 3.2. As in [11, Section 4] for the simplified Ericksen-Leslie model ($\sigma_Q \equiv 0$, $\omega_Q \equiv 0$, $\lambda_2 = 0$), it is also possible to consider the situation where the operators A_0 and B_0 in the general three-parameter family of regularized models represented by problem (2.2) have values from a convergent (in a certain sense) sequence, and study the limiting behavior of the corresponding sequence of energy solutions. As a special case this includes the $\alpha \rightarrow 0^+$ limits in the α -models (1.1). We leave the details for future contributions.

3.2. Regularity of weak solutions. In this subsection, we develop a regularity result for the energy solutions of the general family of regularized models constructed in Section 3.1. Recall that $\theta, \theta_2 \geq 0$, $\theta_1 \in \mathbb{R}$ and that, in general, $\lambda_2 \neq 0$.

Theorem 3.3. *Let*

$$(u, d) \in L^\infty(0, T; \mathcal{Y}_{\theta_2}) \cap L^2(0, T; V^{\theta-\theta_2} \times D(A_1))$$

be an energy solution in the sense of Definition 2.3. Let $s \in (\frac{n}{4}, 1]$, $n = 2, 3$ and consider the following nonempty interval

$$J_n := \left(-\theta_2, \theta - \frac{n}{2}\right) \cap [4s - \theta - 3\theta_2, +\infty).$$

For $\beta \in J_n \neq \emptyset$, let the following conditions hold.

- (i) $b_0 : V^\alpha \times V^\alpha \times V^{\theta-\beta} \rightarrow \mathbb{R}$ is bounded, where $\alpha = \min\{\beta, \theta - \theta_2\}$;
- (ii) $b_0(v, w, Qw) = 0$ for any $v, w \in \mathcal{V}$;
- (iii) $u_0 \in V^\beta$, $d_0 \in D(A_1^s)$, and $g \in L^2(0, T; V^{\beta-\theta})$.

Then we have

$$(3.20) \quad (u, d) \in L^\infty(0, T; V^\beta \times D(A_1^s)) \cap L^2(0, T; V^{\beta+\theta} \times D(A_1^{(2s+1)/2}))$$

and

$$(3.21) \quad \begin{aligned} & \|u(t)\|_\beta^2 + \|d(t)\|_{2s}^2 + \int_0^t \left(\|u(s)\|_{\theta+\beta}^2 + \|d(s)\|_{2s+1}^2 \right) ds \\ & \leq \varphi(t) \left(\|u_0\|_\beta^2 + \|d_0\|_{2s}^2 + \|g\|_{L^2(0, T; V^{\beta-\theta})}^2 \right), \end{aligned}$$

for some positive function φ which depends on time, the norm of the initial data (u_0, d_0) in $V^\beta \times D(A_1^s)$ and on g .

Proof. The following estimates can be rigorously justified working with a sufficiently smooth approximating solution, see Theorem 3.2. We will proceed formally. Pairing the first equation of (2.2) with $\Lambda^{2\beta}u$ yields

$$(3.22) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \langle u, \Lambda^{2\beta}u \rangle + \langle A_0 u, \Lambda^{2\beta}u \rangle + b_0(u, u, \Lambda^{2\beta}u) \\ & = \langle R_0(A_1 d, d), \Lambda^{2\beta}u \rangle + \langle g, \Lambda^{2\beta}u \rangle + \langle \operatorname{div}(\sigma_Q), \Lambda^{2\beta}u \rangle. \end{aligned}$$

Similarly, taking the inner product of the second equation of (2.2) with $A_1^{2s}d$ gives

$$(3.23) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|A_1^s d\|_{L^2}^2 - \frac{1}{\lambda_1} \|A_1^{(2s+1)/2} d\|_{L^2}^2 \\ & = -\langle B_1(u, d), A_1^{2s}d \rangle + \langle \omega_Q d, A_1^{2s}d \rangle + \frac{1}{\lambda_1} \langle f(d), A_1^{2s}d \rangle \\ & \quad - \frac{\lambda_2}{\lambda_1} \langle A_Q d, A_1^{2s}d \rangle. \end{aligned}$$

First, we are going to estimate the b_0 -term as well as all the other terms on the right-hand side of (3.22), then we'll estimate the terms on the right-hand side of (3.23). Combining the boundedness of b_0 (see (i)) with the definition of $\Lambda^{2\beta}u$ and Young's inequality, we find

$$(3.24) \quad b_0(u, u, \Lambda^{2\beta}u) \lesssim \delta^{-1} \|u\|_{\theta-\theta_2}^2 \|u\|_\beta^2 + \delta \|u\|_{\beta+\theta}^2, \quad \text{a.e. in } (0, T),$$

for any $\delta > 0$; clearly, we also have

$$(3.25) \quad \langle g, \Lambda^{2\beta}u \rangle \lesssim \delta \|u\|_{\beta+\theta}^2 + C_\delta \|g\|_{\beta-\theta}^2.$$

Using a duality argument, we get

$$\begin{aligned} |\langle R_0(A_1 d, d), \Lambda^{2\beta} u \rangle| &\lesssim \|R_0(A_1 d, d)\|_{\beta-\theta} \|\Lambda^{2\beta} u\|_{\theta-\beta} \\ &\lesssim \|R_0(A_1 d, d)\|_{-1} \|u\|_{\theta+\beta} \end{aligned}$$

since $\beta - \theta < -1$ (in all space dimensions, for $\beta \in J_n$). Now, by Hahn-Banach theorem and Hölder's inequality,

$$\begin{aligned} \|R_0(A_1 d, d)\|_{-1} &= \sup_{\varphi} \langle A_1 d \cdot \nabla d, \varphi \rangle, \quad \varphi \in W^1, \|\varphi\|_1 = 1, \\ &= \sup_{\varphi} \langle A_1^s d, A_1^{1-s}(\nabla d \varphi) \rangle \\ &\leq \sup_{\varphi} \|A_1^s d\|_{L^2} \|A_1^{1-s}(\nabla d \cdot \varphi)\|_{L^2} \\ &\leq \|A_1^s d\|_{L^2} \|d\|_2 \end{aligned}$$

so that the preceding inequality becomes

$$(3.26) \quad |\langle R_0(A_1 d, d), \Lambda^{2\beta} u \rangle| \lesssim \delta \|u\|_{\theta+\beta}^2 + C_\delta \|A_1^s d\|_{L^2}^2 \|A_1 d\|_{L^2}^2.$$

It remains to estimate the term involving σ_Q . To this end, we note the identity

$$\begin{aligned} (3.27) \quad \langle \operatorname{div}(\sigma_Q), \Lambda^{2\beta} u \rangle &= \mu_1 \langle \operatorname{div}((d^T A_Q d)(d \otimes d)), \Lambda^{2\beta} u \rangle \\ &\quad + \langle \operatorname{div}(\mu_2 \mathcal{N}_Q \otimes d + \mu_3 d \otimes \mathcal{N}_Q), \Lambda^{2\beta} u \rangle \\ &\quad + \langle \operatorname{div}(\mu_5 A_Q d \otimes d + \mu_6 d \otimes A_Q d), \Lambda^{2\beta} u \rangle \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For these nonlinear terms, bounds are derived employing Lemma 7.3 (Appendix), as follows:

- (a) The terms $(d^T A_Q d)_{ij} (d \otimes d)_{jk}$ are a product of functions in L^2 and H^1 and therefore bounded in $H^{\beta-\theta+1}$ since $\beta < \theta - n/2$. Moreover, the terms $d_i d_j$ are a product of functions in H^{2s} and H^1 , and therefore bounded in H^1 since, by assumption, $2s > n/2$.
- (b) To estimate all the nonlinear terms I_2, I_3 , we have to estimate terms of the form $(A_Q d)_i d_j$, $d_k (A_Q d)_l$, which are a product of functions in L^2 and H^{2s} (respectively, in H^{2s} and L^2), and therefore bounded in $H^{\beta-\theta+1}$ provided that $\beta < 2s + \theta - 1 - n/2$. On the other hand, we have to estimate terms of the form $(\mathcal{N}_Q)_i d_j$, $d_k (\mathcal{N}_Q)_l$, which are a product of functions in L^2 and H^{2s} (respectively, in H^{2s} and L^2) and therefore are also bounded in $H^{\beta-\theta+1}$ for $\beta \in J_n$.

By a duality argument, (a) and Young's inequality, it follows

$$\begin{aligned} (3.28) \quad |I_1| &\lesssim \mu_1 \|(d^T A_Q d)(d \otimes d)\|_{\beta-\theta+1} \|u\|_{\beta+\theta} \\ &\lesssim \mu_1 \|d^T A_Q d\|_{L^2} \|d \otimes d\|_1 \|u\|_{\beta+\theta} \\ &\lesssim C_\delta (\mu_1 \|d^T A_Q d\|_{L^2})^2 (\|d\|_{2s} \|d\|_1)^2 + \delta \|u\|_{\beta+\theta}^2. \end{aligned}$$

Using a duality argument once more and exploiting (b), we immediately get

$$(3.29) \quad |I_2| + |I_3| \lesssim C_\delta (\|\mathcal{N}_Q\|_{L^2}^2 + \|A_Q d\|_{L^2}^2) \|d\|_{2s}^2 + \delta \|u\|_{\beta+\theta}^2.$$

Inserting (3.24)-(3.26) and (3.28)-(3.31) into (3.22), then using the coercitivity of A_0 , we derive

$$\begin{aligned} (3.30) \quad &\frac{1}{2} \frac{d}{dt} \langle u, \Lambda^{2\beta} u \rangle + c_{A_0} \|u\|_{\theta+\beta}^2 \\ &\leq 5\delta \|u\|_{\beta+\theta}^2 + C_\delta (\|u\|_{\theta-\theta_2}^2 \|u\|_{\beta}^2 + \|g\|_{\beta-\theta}^2) + C_\delta \|A_1^s d\|_{L^2}^2 \|A_1 d\|_{L^2}^2 \\ &\quad + C_\delta (\mu_1 \|d^T A_Q d\|_{L^2}^2) \|d\|_1^2 \|d\|_{2s}^2 + C_\delta (\|\mathcal{N}_Q\|_{L^2}^2 + \|A_Q d\|_{L^2}^2) \|d\|_{2s}^2. \end{aligned}$$

We now turn to estimating the right-hand side of (3.23). First, we note the identity

$$\begin{aligned} &-\langle B_1(u, d), A_1^{2s} d \rangle + \langle \omega_Q d, A_1^{2s} d \rangle + \frac{1}{\lambda_1} \langle f(d), A_1^{2s} d \rangle - \frac{\lambda_2}{\lambda_1} \langle A_Q d, A_1^{2s} d \rangle \\ &= -\langle A_1^{(2s-1)/2} B_1(u, d), A_1^{(2s+1)/2} d \rangle + \langle A_1^{(2s-1)/2} (\omega_Q d), A_1^{(2s+1)/2} d \rangle \\ &\quad + \frac{1}{\lambda_1} \langle A_1^{(2s-1)/2} f(d), A_1^{(2s+1)/2} d \rangle - \frac{\lambda_2}{\lambda_1} \langle A_1^{(2s-1)/2} (A_Q d), A_1^{(2s+1)/2} d \rangle \end{aligned}$$

$$= J_1 + \dots + J_4.$$

For these terms, bounds are derived employing Lemma 7.3 (Appendix), as follows:

- (c) The terms $(Qu)_i \partial_i d_j$ are a product of functions in $H^{(\beta+\theta+3\theta_2)/2}$ and H^s and therefore bounded in H^{2s-1} provided that $\beta \geq 4s-2-\theta-3\theta_2$ and $\beta > n+2s-2-\theta-3\theta_2$, which are satisfied if $\beta \in J_n$.
- (d) Finally, we have to estimate terms of the form $(\nabla Qu)_{ij} d_j$, which are a product of functions in $H^{(\beta+\theta+3\theta_2)/2-1}$ and H^{s+1} , and therefore bounded in H^{2s-1} provided that $\beta \geq 4s-\theta-3\theta_2$ and $\beta > n+2s-2-\theta-3\theta_2$, which once again holds for $\beta \in J_n$.

We begin with an easy bound on J_3 since $f(d) = (|d|^2 - 1)d$. We have

$$(3.31) \quad \begin{aligned} |J_3| &\lesssim \|A_1^{1/2}(f(d))\|_{L^2} \|A_1^{(2s+1)/2} d\|_{L^2} \\ &\leq \delta \|d\|_{2s+1}^2 + C_\delta \|A_1 d\|_{L^2}^2 \|A_1^s d\|_{L^2}^2. \end{aligned}$$

By an interpolation inequality in the triple $W^{2s+1} \subset W^{s+1} \subset W^1$, Holder and Young inequalities, in view of (c) we find

$$(3.32) \quad \begin{aligned} |J_1| &\lesssim \|B_1(u, d)\|_{2s-1} \|d\|_{2s+1} \\ &\lesssim \|Qu\|_{(\beta+\theta+3\theta_2)/2} \|d\|_{s+1} \|d\|_{2s+1} \\ &\lesssim \|u\|_{(\beta+\theta-\theta_2)/2} \|d\|_1^{1/2} \|d\|_{2s+1}^{3/2} \\ &\leq \delta \|d\|_{2s+1}^2 + C_\delta \|u\|_{(\beta+\theta-\theta_2)/2}^4 \|d\|_1^2 \\ &\leq \delta \|d\|_{2s+1}^2 + C_\delta \|u\|_\beta^2 \|u\|_{\theta-\theta_2}^2 \|d\|_1^2. \end{aligned}$$

Similar to the bound for J_1 , using (d) one deduces

$$(3.33) \quad \begin{aligned} |J_2| &\lesssim \|\omega_Q d\|_{2s-1} \|d\|_{2s+1} \\ &\lesssim \|Qu\|_{(\beta+\theta+3\theta_2)/2} \|d\|_{s+1} \|d\|_{2s+1} \\ &\leq \delta \|d\|_{2s+1}^2 + C_\delta \|u\|_\beta^2 \|u\|_{\theta-\theta_2}^2 \|d\|_1^2. \end{aligned}$$

As a result of (3.32)-(3.33), we also find the same bound (3.33) for J_4 . Putting all the above estimates (3.31)-(3.33) together with (3.23), we arrive at the inequality

$$(3.34) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \|A_1^s d\|_{L^2}^2 - \frac{1}{\lambda_1} \|A_1^{(2s+1)/2} d\|_{L^2}^2 \\ &\leq C_\delta \|u\|_\beta^2 \|u\|_{\theta-\theta_2}^2 \|d\|_1^2 + 4\delta \|d\|_{2s+1}^2 + C_\delta \|A_1 d\|_{L^2}^2 \|A_1^s d\|_{L^2}^2. \end{aligned}$$

Finally, combining (3.30) and (3.34), we infer

$$(3.35) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|A_1^s d\|_{L^2}^2 + \langle u, \Lambda^{2\beta} u \rangle \right] - \frac{1}{\lambda_1} \|A_1^{(2s+1)/2} d\|_{L^2}^2 + c_{A_0} \|u\|_{\theta+\beta}^2 \\ &\leq 5\delta \|u\|_{\beta+\theta}^2 + 4\delta \|d\|_{2s+1}^2 + C_\delta \|u\|_\beta^2 \|u\|_{\theta-\theta_2}^2 \|d\|_1^2 + C_\delta \|A_1 d\|_{L^2}^2 \|A_1^s d\|_{L^2}^2 \\ &\quad + C_\delta \left(\|u\|_{\theta-\theta_2}^2 \|u\|_\beta^2 + \|g\|_{\beta-\theta}^2 \right) + C_\delta \left(\|\mathcal{N}_Q\|_{L^2}^2 + \|A_Q d\|_{L^2}^2 \right) \|d\|_{2s}^2 \\ &\quad + C_\delta (\mu_1 \|d^T A_Q d\|_{L^2})^2 \|d\|_1^2 \|d\|_{2s}^2. \end{aligned}$$

Thus, choosing a sufficiently small $\delta \sim \min(c_{A_0}, -\lambda_1^{-1}) > 0$ in (3.35), by Gronwall's inequality we conclude (3.21). The proof of the theorem is finished. \square

To clarify the previous result in the case of specific models, the corresponding conditions (in particular, (i)-(ii)) when b_0 is either b_{00} or b_{01} , as given by Example 2.2, are listed below in the following remarks. We note that this procedure always produces a new interval \mathcal{Y}_n for the parameter β so that one must ensure that $J_n \cap \mathcal{Y}_n$ stays nonempty.

Remark 3.3. Let $4\theta + 4\theta_1 + 2\theta_2 > n + 2$, $2\theta + 2\theta_1 \geq 1 - k$, $\theta + 2\theta_2 \geq 1$, $3\theta + 4\theta_1 \geq 1$, $\theta + 2\theta_1 \geq \ell$, and $3\theta + 2\theta_1 + 2\theta_2 \geq 2 - \ell$, for some $k, \ell \in \{0, 1\}$. For

$$\beta \in \left(\frac{n+2}{2} - 2(\theta_1 + \theta_2) - \theta, 3\theta + 2\theta_1 - \frac{n+2}{2} \right) \cap \left[\frac{1-\ell}{2} - \theta_1 - \theta_2, \min\{2\theta + \theta_2 - 1, 2\theta - \theta_2 + 2\theta_1 - k\} \right].$$

from [14, Proposition 2.5] we infer that the trilinear form b_{00} satisfies the hypotheses (i)-(ii) of the above theorem.

Remark 3.4. Let $4\theta + 4\theta_1 + 2\theta_2 > n + 2$, $\theta + 2\theta_2 \geq 0$, and $\theta + 2\theta_1 \geq 1$. For

$$\beta \in \left(\frac{n+2}{2} - 2(\theta_1 + \theta_2) - \theta, 3\theta + 2\theta_1 - \frac{n+2}{2} \right) \cap \left[\frac{1}{2} - \theta_1 - \theta_2, \min\{2\theta + \theta_2, 2\theta - \theta_2 + 2\theta_1 - 1\} \right].$$

from [14, Proposition 2.5] it follows that the trilinear form b_{01} satisfies the hypotheses (i)-(ii) of the above theorem.

Remark 3.5. We note that the interval J_n is exactly the same for any fixed values of θ, θ_2 . This is the case, for instance, when $\theta = \theta_2 = 1$, refer to Table 1. We also observe that for $J_n \neq \emptyset$, we must always ask that $\theta + \theta_2 > n/2$.

Remark 3.6. For any $s \in (0.75, 0.875)$ such that $\beta \in [4s - 4, -0.5]$, Theorem 3.3 implies global regularity of the energy solutions of Definition 2.3 for the modified Leray-EL- α (ML-EL- α) model, the SBM-EL model and the NS-EL- α system in three space dimensions. We emphasize that $J_3 = \emptyset$ when $s = 1$ for all these models, and that these results are valid without any restrictions on the physical parameters $\lambda_1, \lambda_2, \mu_1, \dots, \mu_6$, other than what was already assumed in Section 2 (cf. (2.18)-(2.21) and **Cases 1-2**). On the other hand, global regularity of the energy solutions for the 3D Leray-EL- α ($\theta = 1, \theta_2 = 0$) model and the 3D NSV-EL model ($\theta = 0, \theta_2 = 1$) are not covered here since both models fail to satisfy the condition $\theta + \theta_2 > n/2$.

Remark 3.7. Any regular weak solution (u, d) , as given by Theorem 3.3, satisfies

$$d \in L^\infty(0, T; L^\infty(\Omega))$$

due to the Sobolev embedding $W^{2s} \subset L^\infty$, as $2s > n/2$.

3.3. Uniqueness and stability. Now we shall provide sufficient conditions for uniqueness and continuous dependence on the initial data for any weak solutions of the general three-parameter family of regularized models. Recall that $\theta_1 \in \mathbb{R}$ and $\theta, \theta_2 \geq 0$.

Our first result is concerned with the case when a maximum principle applies to the director field d (i.e., when $\lambda_2 = 0$, such that any stretching of the crystal molecules is ignored).

Theorem 3.4. *Let $(u_i, d_i) \in L^\infty(0, T; \mathcal{Y}_{\theta_2})$, $i = 1, 2$, be two energy solutions in the sense of Definition 2.3, corresponding to the initial conditions $(u_i(0), d_i(0)) \in \mathcal{Y}_{\theta_2}$, $i = 1, 2$. Assume Proposition 3.1 and the following conditions.*

(i) $b_0 : V^{\sigma_1} \times V^{\theta - \theta_2} \times V^{\sigma_2} \rightarrow \mathbb{R}$ is bounded for some $\sigma_1 \leq \theta - \theta_2$ and $\sigma_2 \leq \theta + \theta_2$ with $\sigma_1 + \sigma_2 \leq \theta$.

(ii) $b_0(v, w, Qw) = 0$ for any $v \in V^{\sigma_1}$ and $w \in V^{\sigma_2}$.

Further suppose that $\theta_2 \geq 1$. Then the following estimate holds

$$\begin{aligned} (3.36) \quad & \|u_1(t) - u_2(t)\|_{-\theta_2}^2 + \|d_1(t) - d_2(t)\|_1^2 \\ & + \int_0^t \left(\|u_1(s) - u_2(s)\|_{\theta - \theta_2}^2 + \|A_1(d_1(s) - d_2(s))\|_{L^2}^2 \right) ds \\ & \leq \varrho(t) \left(\|u_1(0) - u_2(0)\|_{-\theta_2}^2 + \|d_1(0) - d_2(0)\|_1^2 \right), \end{aligned}$$

for $t \in [0, T]$, for some positive continuous function $\varrho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varrho(0) > 0$, which depends only on the initial data $(u_i(0), d_i(0))$ in \mathcal{Y}_{θ_2} -norm.

Proof. First of all, when $\lambda_2 = 0$ by Proposition 3.1 there exists a constant $M > 0$ such that $\|d_i\|_{L_t^\infty L_x^\infty} \leq M$, $i = 1, 2$. Set $\nabla_d W(d) = f(d)$ and let $u = u_1 - u_2$, $d = d_1 - d_2$. Then subtracting the equations for (u_1, d_1) and (u_2, d_2) we have

$$\begin{aligned} (3.37) \quad & \langle \partial_t u, w \rangle + \langle A_0 u, w \rangle + \langle B_0(u, u_1), w \rangle + \langle B_0(u_2, u), w \rangle \\ & = \langle R_0(A_1 d_2, d), w \rangle + \langle R_0(A_1 d, d_1), w \rangle + \langle \sigma_{Qu_1} - \sigma_{Qu_2}, \nabla w \rangle, \end{aligned}$$

and

$$\begin{aligned} (3.38) \quad & \langle \partial_t d, \eta \rangle - \frac{1}{\lambda_1} \langle A_1 d, \eta \rangle + \langle B_1(u, d_1), \eta \rangle + \langle B_1(u_2, d), \eta \rangle \\ & = \frac{1}{\lambda_1} \langle f(d_1) - f(d_2), \eta \rangle + \langle \omega_{Qu} d_1 + \omega_{Qu_2} d, \eta \rangle. \end{aligned}$$

Here, we denote σ_{Qu_i} , A_{Qu_i} , ω_{Qu_i} to be exactly the same stress tensors from (1.5), (1.3) associated with a given weak solution $u = u_i$, $i = 1, 2$, while $\omega_{Qu} := \omega_{Qu_1} - \omega_{Qu_2}$. First, observe that by the assumptions on θ, θ_2 , according to the estimates that we will perform below, the

weak solution (u_i, d_i) of (2.2) enjoys in fact *additional* regularity. Indeed, these subsequent estimates yield that $R_0(A_1 d_i, d_i)$, $B_0(u_i, u_i) \in L^2(0, T; V^{-\theta-\theta_2})$ and $\sigma_{Qu_i} \in L^2(0, T; V^{-\theta-\theta_2+1})$, and $B_1(u_i, d_i)$, $\omega_{Qu_i} d_i \in L^2(0, T; L^2(\Omega))$. This regularity effectively translates to regularity of the time derivatives $\partial_t u \in L^2(0, T; V^{-\theta-\theta_2})$ and $\partial_t d \in L^2(0, T; L^2(\Omega))$ such that each of the corresponding functional pairings $\langle \partial_t u, w \rangle$ and $\langle \partial_t d, \eta \rangle$ is integrable for $w \in L^2(0, T; V^{\theta+\theta_2})$ and $\eta \in L^2(0, T; L^2(\Omega))$, respectively. Thus, in what follows we can take $w = Qu$ and $\eta = A_1 d$ into (3.37)-(3.38) to infer

$$\begin{aligned}
 (3.39) \quad & \frac{d}{dt} \left(\|u\|_{-\theta_2}^2 + \|A_1^{1/2} d\|_{L^2}^2 \right) + 2c_{A_0} \|u\|_{\theta-\theta_2}^2 - \frac{2}{\lambda_1} \|A_1 d\|_{L^2}^2 \\
 & \leq 2|b_0(u, u_1, Qu)| + 2|b_1(u, d, A_1 d_2)| + 2|b_1(u_2, d, A_1 d)| + \frac{2}{\lambda_1} |\langle f(d_1) - f(d_2), A_1 d \rangle| \\
 & + 2|\langle \omega_{Qu} d_1 + \omega_{Qu_2} d, A_1 d \rangle| + 2|\langle \sigma_{Qu_1} - \sigma_{Qu_2}, \nabla(Qu) \rangle| \\
 & =: I_1 + I_2 + \dots + I_6.
 \end{aligned}$$

All the terms $I_1 - I_4$ on the right-hand side of (3.39) were estimated in [11, Theorem 3.4] for the corresponding regularized simplified Ericksen-Leslie system (2.2) when $\sigma_Q \equiv 0$ and $\omega_Q \equiv 0$. The bounds¹ for these nonlinear terms read as follows:

$$(3.40) \quad \begin{cases} I_1 \lesssim C_\delta \|u\|_{-\theta_2}^2 \|u_1\|_{\theta-\theta_2}^2 + \delta \|u\|_{\theta-\theta_2}^2, \\ I_2 \lesssim \delta \|A_1 d\|_{L^2}^2 + C_\delta \left(\|d\|_1^2 + \|u\|_{-\theta_2}^2 \|A_1 d_2\|_{L^2}^2 \right), \\ I_3 \lesssim \delta \|A_1 d\|_{L^2}^2 + C_\delta \|u_2\|_{\theta-\theta_2}^2 \|u_2\|_{-\theta_2}^\kappa \|d\|_1^2, \\ I_4 \lesssim \delta \|A_1 d\|_{L^2}^2 + C_\delta \|d\|_{L^2}^2, \end{cases}$$

for any $\delta > 0$, for some $\kappa = \kappa(n) \geq 2$ and $C_\delta > 0$ sufficiently large. Now we proceed to estimate I_5 and I_6 . This is mainly the place where the main condition $\theta_2 \geq 1$ (in all space dimensions) must be enforced. We begin with the most challenging term I_6 . With the following definitions

$$\begin{aligned}
 \widehat{\mathcal{N}} &:= \frac{1}{\lambda_1} (A_1 d + (f(d_1) - f(d_2)) d_1 + f(d_2) d), \\
 \mathcal{N}_i &:= \dot{d}_i - \omega_{Qu_i} d_i,
 \end{aligned}$$

we note the following identity

$$\begin{aligned}
 (3.41) \quad & \langle \sigma_{Qu_1} - \sigma_{Qu_2}, \nabla(Qu) \rangle = \mu_1 \langle (d_1^T A_{Qu_1} d_1) (d_1 \otimes d_1) - (d_2^T A_{Qu_2} d_2) (d_2 \otimes d_2), \nabla Qu \rangle \\
 & + \mu_2 \langle \mathcal{N}_1 \otimes d_1 - \mathcal{N}_2 \otimes d_2, \nabla Qu \rangle + \mu_3 \langle d_1 \otimes \mathcal{N}_1 - d_2 \otimes \mathcal{N}_2, \nabla Qu \rangle \\
 & + \mu_5 \langle (A_{Qu_1} d_1) \otimes d_1 - (A_{Qu_2} d_2) \otimes d_2, \nabla Qu \rangle \\
 & + \mu_6 \langle d_1 \otimes (A_{Qu_1} d_1) - d_2 \otimes (A_{Qu_2} d_2), \nabla Qu \rangle \\
 & = \mu_1 \langle (d_1^T A_{Qu_1} d_1 + d_2^T A_{Qu_1} d_1 + d_2^T A_{Qu_2} d_2) (d_1 \otimes d_1), \nabla Qu \rangle \\
 & + \mu_1 \langle (d_2^T A_{Qu_2} d_2) (d \otimes d_1 + d_2 \otimes d), \nabla Qu \rangle \\
 & + \mu_2 \langle \widehat{\mathcal{N}} \otimes d_1 + \mathcal{N}_2 \otimes d, \nabla Qu \rangle + \mu_3 \langle d_1 \otimes \widehat{\mathcal{N}} + d \otimes \mathcal{N}_2, \nabla Qu \rangle \\
 & + \mu_5 \langle (A_{Qu} d_1 + A_{Qu_2} d) \otimes d_1 + (A_{Qu_2} d_2) \otimes d, \nabla Qu \rangle \\
 & + \mu_6 \langle d_1 \otimes (A_{Qu} d_1 + A_{Qu_2} d) + d \otimes (A_{Qu_2} d_2), \nabla Qu \rangle \\
 & =: I_{61} + I_{62} + \dots + I_{66}.
 \end{aligned}$$

We shall estimate $I_{61} - I_{66}$ now. By the Holder inequality and proper Sobolev embedding theorems (e.g., $V^{-\theta_2} \subseteq V^{1-2\theta_2}$ and $W^2 \subset L^\infty$), we have

$$\begin{aligned}
 (3.42) \quad & |I_{61}| \lesssim (\|d\|_{L^\infty} \|\nabla Qu_1\|_0 \|d_1\|_{L^\infty} + \|d_2\|_{L^\infty} \|\nabla Qu\|_0 \|d_1\|_{L^\infty} + \|d_2\|_{L^\infty} \|\nabla Qu_2\|_0 \|d\|_{L^\infty}) \\
 & \times \|d_1\|_{L^\infty}^2 \|\nabla Qu\|_0 \\
 & \leq C_M \left(\|d\|_{L^\infty} \|u_1\|_{-\theta_2} \|u\|_{-\theta_2} + \|u\|_{-\theta_2}^2 + \|u_2\|_{-\theta_2} \|d\|_{L^\infty} \|u\|_{-\theta_2} \right) \\
 & \leq C_M \|u\|_{-\theta_2}^2 + \delta \|A_1 d\|_{L^2}^2 + C_{M,\delta} \left(\|u_1\|_{-\theta_2}^2 + \|u_2\|_{-\theta_2}^2 \right) \|u\|_{-\theta_2}^2
 \end{aligned}$$

¹The estimates from (3.40) performed in [11] required that $\theta + \theta_2 \geq 1$, $\theta_2 \geq 0$ in 2D and $\theta_2 \geq 1$ in 3D. Alternatively, one can replace these conditions by $\theta + \theta_2 > \frac{n}{2}$ which is complementary.

for any $\delta > 0$, since $\|d_i\|_{L_t^\infty L_x^\infty} \leq M$, $i = 1, 2$. Similarly, we have

$$(3.43) \quad \begin{aligned} |I_{62}| &\lesssim \|d\|_{L^\infty} \left(\|d_2\|_{L^\infty}^2 \|d_1\|_{L^\infty} + \|d_2\|_{L^\infty}^3 \right) \|\nabla Q u_2\|_0 \|\nabla Q u\|_0 \\ &\leq C_{M,\delta} \|u_2\|_{-\theta_2}^2 \|u\|_{-\theta_2}^2 + \delta \|A_1 d\|_{L^2}^2. \end{aligned}$$

Moreover, recalling that $\mathcal{N}_i \in L^2(0, T; L^2(\Omega))$, $i = 1, 2$, for every weak energy solution (u_i, d_i) of Definition 2.3, we estimate

$$(3.44) \quad \begin{aligned} |I_{63}| &\leq C_M (\|A_1 d\|_{L^2} + \|d\|_{L^2}) \|\nabla Q u\|_0 + \|\mathcal{N}_2\|_{L^2} \|d\|_{L^\infty} \|\nabla Q u\|_0 \\ &\leq \delta \|A_1 d\|_{L^2}^2 + C_{M,\delta} \|u\|_{-\theta_2}^2 \left(1 + \|\mathcal{N}_2\|_{L^2}^2 \right). \end{aligned}$$

The bound for I_{64} is exactly the same as in (3.44). On the other hand, the bound for the last integrals I_{65}, I_{66} can be obtained as follows:

$$(3.45) \quad \begin{aligned} \max\{|I_{65}|, |I_{66}|\} &\leq C_M \left(\|\nabla Q u\|_0^2 + \|\nabla Q u_2\|_0 \|d\|_{L^\infty} \|\nabla Q u\|_0 \right) \\ &\leq \delta \|A_1 d\|_{L^2}^2 + C_{M,\delta} \|u\|_{-\theta_2}^2 \left(1 + \|u_2\|_{-\theta_2}^2 \right). \end{aligned}$$

To estimate I_5 , we start with the preliminary estimate

$$\begin{aligned} |\langle \omega_{Qu} d_1, A_1 d \rangle| &\lesssim \|d_1\|_{L^\infty} \|A_1 d\|_{L^2} \|\nabla Q u\|_0 \\ &\leq \delta \|A_1 d\|_{L^2}^2 + C_{M,\delta} \|u\|_{-\theta_2}^2. \end{aligned}$$

Next, by Agmon inequalities we have

$$\begin{aligned} |\langle \omega_{Qu_2} d, A_1 d \rangle| &\lesssim \|d\|_{L^\infty} \|\nabla Q u_2\|_0 \|A_1 d\|_{L^2} \\ &\lesssim \begin{cases} \|d\|_1^{1/2} \|A_1 d\|_{L^2}^{3/2} \|u_2\|_{-\theta_2}, & \text{if } n = 2, \\ \|d\|_1^{1/4} \|A_1 d\|_{L^2}^{7/4} \|u_2\|_{-\theta_2}, & \text{if } n = 3. \end{cases} \end{aligned}$$

Therefore, using Young's inequality, it follows

$$(3.46) \quad |I_5| \leq \delta \|A_1 d\|_{L^2}^2 + C_\delta \left[\|u\|_{-\theta_2}^2 + \left(\|u_2\|_{-\theta_2}^4 + \|u_2\|_{-\theta_2}^8 \right) \|d\|_1^2 \right].$$

Combining (3.40), (3.42)-(3.46), then choosing a sufficiently small $\delta \sim \min(c_{A_0}, -\lambda_1^{-1}) > 0$ into (3.39), by application of Gronwall's inequality, one finds

$$\|u(t)\|_{-\theta_2}^2 + \|d(t)\|_1^2 \leq \left(\|u(0)\|_{-\theta_2}^2 + \|d(0)\|_1^2 \right) \exp \int_0^t \Theta(s) ds,$$

for a suitable function $\Theta \in L^1(0, T)$. Integrating (3.39) once more over $(0, t)$ yields the desired inequality (3.36). The proof is finished. \square

To clarify these stability results at least in the case of the specific models listed in Table 1, the corresponding conditions and stability results derived from Theorem 3.4 are given below.

Remark 3.8. Exploiting [14, Proposition 2.5], the trilinear form b_{00} satisfies the hypotheses of Theorem 3.4 provided $\theta + \theta_1 \geq \frac{1-k}{2}$, $\theta + 2\theta_1 \geq k$, $\theta + \theta_2 \geq \frac{1}{2}$, $2\theta + 2\theta_1 + \theta_2 > \frac{n+2}{2}$, and $3\theta + 2\theta_1 + 2\theta_2 \geq 2 - k$, for some $k \in \{0, 1\}$. The trilinear form b_{01} satisfies the hypotheses of Theorem 3.4 for $\theta + 2\theta_1 \geq 1$, $\theta + \theta_1 \geq \frac{1}{2}$, $\theta + \theta_2 \geq 0$, $2\theta + 2\theta_1 + \theta_2 > \frac{n+2}{2}$, and $3\theta + 2\theta_1 + 2\theta_2 \geq 1$. Together with $\theta_2 \geq 1$ and $\lambda_2 = 0$, these assumptions allow us to recover the stability and uniqueness of energy solutions for the 3D NS-EL- α -model, the 3D NSV-EL- α -model, the 3D ML-EL- α -model and the 3D SBM-EL model (see Table 1). These results were not reported anywhere else.

We conclude the section with a result that handles the general case when $\lambda_2 \neq 0$.

Theorem 3.5. *Let*

$$(u_i, d_i) \in L^\infty(0, T; V^\beta \times D(A_1^s)) \cap L^2(0, T; V^{\beta+\theta} \times D(A_1^{(2s+1)/2}))$$

be two energy solutions that satisfy the assumptions of Theorem 3.3. For $\theta_2 \geq 1$, the estimate

$$(3.47) \quad \begin{aligned} &\|u_1(t) - u_2(t)\|_{-\theta_2}^2 + \|d_1(t) - d_2(t)\|_1^2 \\ &+ \int_0^t \left(\|u_1(s) - u_2(s)\|_{\theta-\theta_2}^2 + \|A_1(d_1(s) - d_2(s))\|_{L^2}^2 \right) ds \end{aligned}$$

$$\leq \varrho(t) \left(\|u_1(0) - u_2(0)\|_{-\theta_2}^2 + \|d_1(0) - d_2(0)\|_1^2 \right),$$

holds for $t \in [0, T]$, for some positive continuous function $\varrho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varrho(0) > 0$, which depends on the initial data $(u_i(0), d_i(0))$ in $V^\beta \times W^{2s}$ -norm.

Proof. Indeed, the main ingredient in the proof of Theorem 3.4 was the fact that $d_i \in L_t^\infty(L_x^\infty)$ which is now provided by Remark 3.7. It is worth pointing out that in the general case when $\lambda_2 \neq 0$, the inequality (3.39) reads as follows:

$$\begin{aligned} (3.48) \quad & \frac{d}{dt} \left(\|u\|_{-\theta_2}^2 + \|A_1^{1/2}d\|_{L^2}^2 \right) + 2c_{A_0}\|u\|_{\theta-\theta_2}^2 - \frac{2}{\lambda_1} \|A_1d\|_{L^2}^2 \\ & \leq 2|b_0(u, u_1, Qu)| + 2|b_1(u, d, A_1d_2)| + 2|b_1(u_2, d, A_1d)| + \frac{2}{\lambda_1} |\langle f(d_1) - f(d_2), A_1d \rangle| \\ & + 2|\langle \omega_{Qu}d_1 + \omega_{Qu_2}d, A_1d \rangle| + 2|\langle \sigma_{Qu_1} - \sigma_{Qu_2}, \nabla(Qu) \rangle| - 2\lambda_2\lambda_1^{-1} |\langle A_{Qu}d_1 + A_{Qu_2}d, A_1d \rangle| \\ & =: I_1 + I_2 + \dots + I_6 + I_7. \end{aligned}$$

More precisely, with respect to (3.39), there is one additional term I_7 on the right-hand side. Bounds on the first six terms I_1 - I_6 are already provided by (3.40)-(3.46). To find a proper bound for the final term I_7 one may proceed *verbatim* as in getting estimate (3.46) for the term I_5 ; indeed, note that A_Q and ω_Q are in fact the symmetric and the skew-symmetric parts of the strain rate, respectively (cf. (1.3)). Hence, the proof of (3.47) follows from that of (3.36) with some minor modifications. \square

4. FINITE DIMENSIONAL GLOBAL ATTRACTORS

In this section we establish the existence of (smooth) finite dimensional global attractors for the general three-parameter family of regularized models (2.2). For the sake of reference below, recall the following definition for the space of translation bounded functions

$$L_{tb}^2(\mathbb{R}_+; X) := \left\{ g \in L_{loc}^2(\mathbb{R}_+; X) : \|g\|_{L_{tb}^2(\mathbb{R}_+; X)}^2 := \sup_{t \geq 0} \int_t^{t+1} \|g(s)\|_X^2 ds < \infty \right\},$$

where X is a given Banach space.

We begin with a first basic dissipative inequality which is satisfied by any weak energy solution of problem (2.2). The following result holds for any $\theta, \theta_2 \geq 0$.

Proposition 4.1. *Let $(u, d) \in L_{loc}^\infty(0, \infty; \mathcal{Y}_{\theta_2}) \cap L_{loc}^2(0, \infty; V^{\theta-\theta_2} \times D(A_1))$ be any energy solution in the sense of Definition 2.4 with $(u(0), d(0)) \in \mathcal{Y}_{\theta_2}$. Let the following conditions hold.*

- (i) $\langle A_0v, Qv \rangle \geq c_{A_0}\|v\|_{\theta-\theta_2}^2$ for any $v \in V^{\theta-\theta_2}$, with a constant $c_{A_0} > 0$;
- (ii) $g \in L_{tb}^2(\mathbb{R}_+; V^{-\theta-\theta_2})$;

Then for some constant $\kappa > 0$ independent of time and the initial condition, we have

$$\begin{aligned} (4.1) \quad & \|u(t)\|_{-\theta_2}^2 + \|d(t)\|_1^2 + \|(u, d)\|_{L^2(t, t+1; V^{\theta-\theta_2} \times D(A_1))}^2 \\ & \lesssim e^{-\kappa t} \left(\|u(0)\|_{-\theta_2}^2 + \mathcal{L}(\|d(0)\|_1^2) \right) + C_*, \end{aligned}$$

for all $t \geq 0$, for some constant $C_* > 0$ and a function $\mathcal{L} > 0$ independent of time and the initial data.

Proof. The proof of estimate (4.1) follows the line of arguments given in [11, Proposition 5.1]. However the arguments in the present case are simpler since f is precisely given. For completeness sake, we include a short proof of this dissipative estimate. As usual, one proves the claim for smooth approximate solutions and then one passes to the limit in the end result. First, we observe that we can find a positive function \mathcal{L} such that

$$(4.2) \quad \|u(t)\|_{-\theta_2}^2 + \|d(t)\|_1^2 \lesssim E_Q(t) \lesssim \|u(t)\|_{-\theta_2}^2 + \mathcal{L}(\|d(t)\|_1^2),$$

owing to the definition of Q , the fact that $W(d) = \frac{1}{4}(|d|^2 - 1)^2$ and standard Sobolev inequalities. Next, let us set $\rho := A_1d + f(d)$ and note that

$$\|A_1^{1/2}d\|_{L^2}^2 + \langle f(d)d, 1 \rangle = \|A_1^{1/2}d\|_{L^2}^2 + \int_\Omega |d|^4 - \int_\Omega |d|^2 = \langle \rho, d \rangle$$

since $f(d) = (|d|^2 - 1)d$. By simple manipulation of Young and Cauchy-Schwarz inequalities, for any $\delta \geq 1/8$, it easily follows

$$\begin{aligned} \frac{1}{2} \|A_1^{1/2} d\|_{L^2}^2 + 2 \langle W(d), 1 \rangle &= \frac{1}{2} \langle \rho, d \rangle - \frac{1}{2} \langle |d|^2 - 1, 1 \rangle \\ &\leq \delta \|\rho\|_{L^2}^2 + \frac{1}{2} \left(\frac{1}{8\delta} - 1 \right) \|d\|_{L^2}^2 + \frac{1}{2} |\Omega| \\ &\leq \delta \|\rho\|_{L^2}^2 + \frac{1}{2} |\Omega|. \end{aligned}$$

Thus, we obtain

$$\frac{1}{2} \|A_1^{1/2} d\|_{L^2}^2 + \langle W(d), 1 \rangle \leq \frac{3\delta}{2} \|\rho\|_{L^2}^2 + C.$$

This relation together with either one of the energy identities (2.25)-(2.26) and the assumption (i), yields for $\kappa \in (0, 1)$, the following inequality

$$(4.3) \quad \frac{d}{dt} E_Q(t) + \kappa E_Q(t) + \mu_1 \|(d^T A_Q d)(t)\|_{L^2}^2 \leq \Theta(t),$$

where we have set

$$\begin{aligned} \Theta(t) &:= -c_{A_0} \|u(t)\|_{\theta-\theta_2}^2 + \kappa c_Q \|u(t)\|_{-\theta_2}^2 + \frac{1}{\lambda_1} \|\rho(t)\|_{L^2}^2 + \tilde{\delta}^{-1} \|g(t)\|_{-\theta-\theta_2}^2 \\ &\quad + \tilde{\delta} \|Q\|_{-\theta_2; \theta_2}^2 \|u(t)\|_{\theta-\theta_2}^2 + \frac{3\kappa\delta}{2} \|\rho(t)\|_{L^2}^2, \end{aligned}$$

for any $\tilde{\delta} > 0$. Setting now $\delta = 2/3$ and $\tilde{\delta} > 0$ in such a way that $\tilde{\delta} \|Q\|_{-\theta_2; \theta_2}^2 = c_{A_0}/2$, and exploiting the Sobolev embedding $V^{\theta-\theta_2} \subseteq V^{-\theta_2}$ (i.e., $\|u\|_{-\theta_2}^2 \leq C_{\theta_2} \|u\|_{\theta-\theta_2}^2$ for $\theta \geq 0$), it follows

$$\Theta(t) \leq -\frac{c_{A_0}}{2} \left(1 - \frac{2}{c_{A_0}} \kappa c_Q C_{\theta_2} \right) \|u(t)\|_{\theta-\theta_2}^2 - (\lambda_1^{-1} - \kappa) \|\rho(t)\|_{L^2}^2 + C(1 + \|g(t)\|_{-\theta-\theta_2}^2),$$

for all $t \geq 0$. Adjusting a sufficiently small constant $\kappa \in (0, \min(-\lambda_1^{-1}/2, c_{A_0}/(4c_Q C_{\theta_2})))$, from (4.3) we infer

$$(4.4) \quad \begin{aligned} \frac{d}{dt} E_Q(t) + \kappa E_Q(t) + \mu_1 \|(d^T A_Q d)(t)\|_{L^2}^2 + C_\kappa (\|\rho(t)\|_{L^2}^2 + \|u(t)\|_{\theta-\theta_2}^2) \\ \lesssim 1 + \|g(t)\|_{-\theta-\theta_2}^2. \end{aligned}$$

The application of Gronwall's inequality (see Appendix, Lemma 7.1) in (4.4) allows us to deduce

$$(4.5) \quad \begin{aligned} E_Q(t) + C_\kappa \int_t^{t+1} (\|u(s)\|_{\theta-\theta_2}^2 + \|\rho(s)\|_{L^2}^2) ds \\ \leq E_Q(0) e^{-\kappa t} + C(1 + \|g\|_{L_{tb}^2(\mathbb{R}_+; V^{-\theta-\theta_2})}), \end{aligned}$$

for all $t \geq 0$, for some positive constants C_κ, C independent of time and the initial data. Reporting (4.2) in (4.5), we easily arrive at the dissipative estimate (4.1). This completes the proof. \square

Next, we recall that by Theorem 3.5, there exists a unique energy solution

$$(u, d) \in L_{\text{loc}}^\infty(0, \infty; \Upsilon_{\beta, s}) \cap L_{\text{loc}}^2(0, \infty; V^{\beta+\theta} \times D(A_1^{(2s+1)/2})),$$

satisfying (2.13)-(2.14) with any given initial data $(u_0, d_0) \in \Upsilon_{\beta, s} := V^\beta \times D(A_1^s)$. Thus, when the body force g is time independent we can define a dynamical system for these regular energy solutions. Indeed, system (2.2) generates a semigroup $\{S_{\theta_2}(t)\}_{t \geq 0}$ of *closed* operators on the Hilbert space $\Upsilon_{\beta, s}$ (when endowed with the metric of $V^{-\theta_2} \times W^1$), given by

$$(4.6) \quad \begin{aligned} S_{\theta_2}(t) : \Upsilon_{\beta, s} &\rightarrow \Upsilon_{\beta, s}, \quad t \geq 0, \\ (u_0, d_0) &\mapsto (u(t), d(t)). \end{aligned}$$

Remark 4.1. In the case $\lambda_2 = 0$, $\mu_1 \geq 0$, by Theorem 3.4 one can also define the dynamical system $(S_{\theta_2}, \mathcal{Y}_{\theta_2})$ for problem (2.2). In this instance in (4.6), $(u(t), d(t))$ is the (unique) energy solution associated with a given initial datum (u_0, d_0) in the space \mathcal{Y}_{θ_2} .

In this section, we will only focus on the *general case* when $\mu_1 \geq 0$ and $\lambda_2 \neq 0$ since the former $\lambda_2 = 0, \mu_1 \geq 0$ is much easier to handle due the validity of the maximum principle for d , cf. Proposition 3.1. The following proposition establishes the existence of an absorbing ball in $\Upsilon_{\beta,s}$ for the dynamical system $(S_{\theta_2}, \Upsilon_{\beta,s})$ in the case $\theta > 0, \theta_2 \geq 1$. Here and everywhere else, $\mathcal{B}_X(R)$ denotes the ball in X of radius R , centered at 0.

Proposition 4.2. *Let $s \in (\frac{n}{4}, 1]$, $n = 2, 3$ and consider the following nonempty interval*

$$J_n := \left(-\theta_2, \min \left(\theta - \frac{n}{2}, \theta - \theta_2 \right) \right] \cap [4s - \theta - 3\theta_2, +\infty).$$

For $\beta \in J_n \setminus \{\theta - n/2\} \neq \emptyset$, let the following conditions hold.

- (i) $b_0 : V^\alpha \times V^\alpha \times V^{\theta-\beta} \rightarrow \mathbb{R}$ is bounded, where $\alpha = \min\{\beta, \theta - \theta_2\}$;
- (ii) $b_0(v, w, Qw) = 0$ for any $v, w \in \mathcal{V}$;
- (iii) $g \in V^{\beta-\theta}$ is time independent.

Then for every $R > 0$, there exists $t_* = t_*(R) > 0$, such that, for any $\varphi_0 := (u_0, d_0) \in \mathcal{B}_{\Upsilon_{\beta,s}}(R)$,

$$(4.7) \quad \sup_{t \geq t_*} \left(\|(u(t), d(t))\|_{\Upsilon_{\beta,s}}^2 + \int_t^{t+1} \left(\|u(s)\|_{\theta+\beta}^2 + \|d(s)\|_{2s+1}^2 \right) ds \right) \leq C,$$

for some constant $C > 0$, independent of time and the initial data.

Proof. By Propositions 4.1, for every $R > 0$ with $\varphi_0 \in \mathcal{B}_{\Upsilon_{\beta,s}}(R)$ there exists $t_0 = t_0(R) > 0$ such that

$$(4.8) \quad \sup_{t \geq t_0} \|(u(t), d(t))\|_{\mathcal{Y}_{\theta_2}}^2 + \int_t^{t+1} \left(\|u(s)\|_{\theta-\theta_2}^2 + \|A_1 d(s)\|_{L^2}^2 \right) ds \leq C_*.$$

Moreover, by Theorem 3.3 and application of the uniform Gronwall's lemma [21, Lemma III.1.1] in (3.35), by virtue of (4.8), we infer the existence of a new time $t_* = t_0 + 1$ such that

$$(4.9) \quad \sup_{t \geq t_*} \left(\|u(t)\|_{\beta}^2 + \|d(t)\|_{2s}^2 \right) \leq C,$$

for some positive constant C independent of time and the initial data. Moreover, integration over $(t, t+1)$ of the inequality (3.35) yields

$$(4.10) \quad \sup_{t \geq t_*} \int_t^{t+1} \left(\|u(s)\|_{\beta+\theta}^2 + \|d(s)\|_{2s+1}^2 \right) ds \leq C,$$

owing once again to (4.9). The claim (4.7) is then immediate. \square

Next we show the existence of finite dimensional global attractors for our regularized family of models (2.2) when $\theta > 0$. However, due to lack of compactness of the solutions in the space $\Upsilon_{\beta,s}$ we cannot proceed in a standard way. Indeed, the strong coupling in the full Ericksen-Leslie system (2.2) for (u, d) prevents us from establishing any additional smoothing properties of the solutions without requiring more restrictive assumptions on the body force g and the other parameters of the problem. In fact, in what follows we shall prove even more: the existence of an exponential attractor for $(S_{\theta_2}, \Upsilon_{\beta,s})$. We recall that the exponential attractor always contains the global attractor and also attracts bounded subsets of the energy phase-space at an exponential rate, which makes it a more useful object in numerical simulations than the global attractor.

We shall accomplish this program in a series of subsequent lemmas. First, we have the basic statement.

Proposition 4.3. *Let \mathcal{B}_0 be a bounded absorbing ball whose existence has been proven in Proposition 4.2. The set*

$$\mathcal{B}_* := \bigcup_{t \geq t_*} S_{\theta_2}(t) \mathcal{B}_0$$

is bounded in $\Upsilon_{\beta,s}$ and positively invariant for S_{θ_2} .

Clearly, \mathcal{B}_* is also absorbing for the semigroup S_{θ_2} . Thus, it is sufficient to construct the exponential attractor for the restriction of this semigroup on \mathcal{B}_* only. With this at hand, we can show the uniform Hölder continuity of $t \mapsto S_{\theta_2}(t) \varphi_0$ in the $V^{-\theta_2} \times W^1$ -norm, namely,

Lemma 4.4. *Let the assumptions of Proposition 4.2 be satisfied. Consider $\varphi(t) = S_{\theta_2}(t) \varphi_0$ with $\varphi_0 \in \mathcal{B}_*$. Then, we have*

$$(4.11) \quad \|u(t) - u(\tilde{t})\|_{-\theta_2} + \|d(t) - d(\tilde{t})\|_1 \lesssim (|t - \tilde{t}|^{\epsilon_1} + |t - \tilde{t}|^{\epsilon_2}),$$

for all $t, \tilde{t} \in [0, T]$, for some $0 < \epsilon_1, \epsilon_2 < 1$ depending only on θ, θ_2 and $s \in (\frac{n}{4}, 1]$.

Proof. We can rely once again on the proof of Theorem 3.2 and Theorem 3.4. Indeed, since the V^β -norm of u and the W^{2s} -norm of d are globally bounded by Theorem 3.3 if $\varphi_0 \in \mathcal{B}_*$, then also

$$\partial_t \varphi = (\partial_t u, \partial_t d) \in L_{\text{loc}}^2(0, \infty; V^{-\theta-\theta_2} \times L^2(\Omega)).$$

Finally, the simple relation

$$\varphi(t) - \varphi(\tilde{t}) = \int_{\tilde{t}}^t \partial_s \varphi(s) ds$$

and proper interpolation inequalities in the spaces $V^\beta \subset V^{-\theta_2} \subset V^{-\theta-\theta_2}$, $W^{2s} \subset W^1 \subset L^2(\Omega)$, $2s > n/2$, imply the desired inequality (4.11). \square

The crucial step in order to establish the existence of an exponential attractor is the validity of so-called smoothing property for the difference of any two energy solutions φ_i , $i = 1, 2$. In the present case, such a property is a consequence of the following two lemmas. The first result establishes that the semigroup $S_{\theta_2}(t)$ is some kind of contraction map, up to the term $\|\varphi_1 - \varphi_2\|_{L^2(0,t;\mathcal{Y}_{\theta_2})}$.

Lemma 4.5. *Let the assumptions of Proposition 4.2 hold. For any two energy solutions $\varphi_i = (u_i, d_i)$ associated with the initial data $\varphi_{0i} \in \mathcal{B}_*$, the following estimate holds:*

$$(4.12) \quad \begin{aligned} & \|u_1(t) - u_2(t)\|_{-\theta_2}^2 + \|d_1(t) - d_2(t)\|_1^2 \\ & \lesssim e^{-\eta t} \left(\|u_1(0) - u_2(0)\|_{-\theta_2}^2 + \|d_1(0) - d_2(0)\|_1^2 \right) \\ & + \int_0^t \left(\|u_1(s) - u_2(s)\|_{-\theta_2}^2 + \|d_1(s) - d_2(s)\|_1^2 \right) ds, \end{aligned}$$

for all $t \geq 0$, for some positive constant η independent of time.

Proof. We rely mainly on the estimates exploited in the proof of Theorem 3.4 and Theorem 3.5. Indeed, each energy solution φ_i is globally bounded in $L^\infty(0, \infty; L^\infty(\Omega))$ by Remark 3.7 if $\varphi_{0i} \in \mathcal{B}_*$. It turns out that the main steps require nothing more than what is already contained in the proof of Theorem 3.4 (or Theorem 3.5).

Our starting point is the inequality (3.48). With the exception of I_2, I_{63}, I_{64} , all the other terms can be estimated word by word as in (3.40), (3.42), (3.43), (3.45) and (3.46), respectively. For I_2, I_{63} and I_{64} , we need more refined estimates. We have the bounds:

$$(4.13) \quad \begin{aligned} |I_2| &= |\langle A_1^{1-s} B_1(u, d), A_1^s d_2 \rangle| \\ &\lesssim \|d_2\|_{2s} \|Qu\|_{\theta_2} \|\nabla d\|_1 \\ &\leq \delta \|A_1 d\|_{L^2}^2 + C \|u\|_{-\theta_2}^2 \|d_2\|_{2s}^2 \end{aligned}$$

and

$$(4.14) \quad \begin{aligned} |I_{63}| &= \left| \mu_2 \langle \hat{\mathcal{N}} \otimes d_1 - \mathcal{N}_2 \otimes d, \nabla Qu \rangle \right| \\ &\lesssim \|\nabla Qu\|_{\theta_2-1} \left(\|\hat{\mathcal{N}} \otimes d_1\|_{1-\theta_2} + \|\mathcal{N}_2 \otimes d\|_{1-\theta_2} \right) \\ &\leq C_M \|u\|_{-\theta_2} (\|A_1 d\|_{L^2} + \|u_2\|_{-\theta_2} \|d_2\|_{2s} \|A_1 d\|_{L^2}) \\ &\leq \delta \|A_1 d\|_{L^2}^2 + C_{\delta, M} \|u\|_{-\theta_2}^2 \left(1 + \|u_2\|_{-\theta_2}^2 \|d_2\|_{2s}^2 \right). \end{aligned}$$

In detail, these bounds are deduced using the definition of $\hat{\mathcal{N}}$ and \mathcal{N}_2 together with the following crucial properties:

- (a1) Each term $Qu_i(\partial_i d_j)$ is a product of functions in H^{θ_2} and H^1 and therefore bounded in H^{2-2s} , by Lemma 7.3 (Appendix) since $2s > n/2$ and $\theta_2 \geq 1$. This yields (4.13).
- (a2) For I_{63} , we observe that each term $[(\partial_i Qu_{2j}) d_j] d_l$ is a product of functions from H^0 and $H^2 \subset L^\infty$ and therefore bounded in $H^0 \subseteq H^{1-\theta_2}$. Finally, the terms $(\partial_i Qu_{2j}) d_j$ are products of functions in H^{θ_2-1} and $H^{2s} \subset L^\infty$, and therefore bounded in L^2 .

A similar argument to the derivation of (4.14) yields the bound

$$|I_{64}| \leq \delta \|A_1 d\|_{L^2}^2 + C_{\delta, M} \|u\|_{-\theta_2}^2 \left(1 + \|u_2\|_{-\theta_2}^2 \|d_2\|_{2s}^2\right).$$

Since also $(u_i, d_i) \in L^\infty(0, \infty; V^\beta \times W^{2s})$, from (3.48) we finally see that

$$\begin{aligned} & \frac{d}{dt} \left(\|u(t)\|_{-\theta_2}^2 + \|d(t)\|_1^2 \right) + (2c_{A_0} - 6\delta) \|u(t)\|_{\theta-\theta_2}^2 + (-2\lambda_1^{-1} - 10\delta) \|A_1 d(t)\|_{L^2}^2 \\ & \leq C_{M, \delta} \left(\|u(t)\|_{-\theta_2}^2 + \|d(t)\|_1^2 \right), \end{aligned}$$

for all $t \geq 0$, for a sufficiently small $\delta \in (0, \min(c_{A_0}/3, -\lambda_1^{-1}/5))$. Thus, Gronwall's inequality entails the desired estimate (4.12). \square

We now need some compactness for the term $\|\varphi_1 - \varphi_2\|_{L^2(0, t; \mathcal{Y}_{\theta_2})}$ on the right-hand side of (4.12). This is given by

Lemma 4.6. *Let the assumptions of Proposition 4.2 hold. Then, the following estimate holds:*

$$\begin{aligned} (4.15) \quad & \|\partial_t u_1 - \partial_t u_2\|_{L^2(0, t; V^{-\theta-\theta_2})}^2 + \|\partial_t d_1 - \partial_t d_2\|_{L^2(0, t; L^2(\Omega))}^2 \\ & + \int_0^t \left(\|u_1(s) - u_2(s)\|_{-\theta_2}^2 + \|d_1(s) - d_2(s)\|_1^2 \right) ds \\ & \leq \varrho(t) \left(\|u_1(0) - u_2(0)\|_{-\theta_2}^2 + \|d_1(0) - d_2(0)\|_1^2 \right), \end{aligned}$$

for all $t \geq 0$.

Proof. The required control of the integral term on the left-hand side of (4.15) is readily provided by (3.47). It remains to gain some control on the time derivative $(\partial_t u, \partial_t d)$, where $u := u_1 - u_2$, $d := d_1 - d_2$. We recall the variational formulation (3.37)-(3.38) and rely once again on the fact that

$$(4.16) \quad \varphi_i = (u_i, d_i) \in L^\infty(0, \infty; \mathcal{B}_*), \quad d_i \in L^\infty(0, \infty; L^\infty(\Omega)),$$

for each $i = 1, 2$. For any test functions $w \in V^{\theta+\theta_2}$ and $\eta \in L^2(\Omega)$, using the corresponding variational formulation, one has

$$\begin{aligned} (4.17) \quad & |\langle \partial_t u, w \rangle| \leq \left(\|\text{r.h.s.u}\|_{-\theta-\theta_2} + \|\sigma_{Qu_1} - \sigma_{Qu_2}\|_{1-\theta-\theta_2} \right) \|w\|_{\theta+\theta_2}, \\ & |\langle \partial_t d, \eta \rangle| \leq \|\text{r.h.s.d}\|_{L^2} \|\eta\|_{L^2}, \end{aligned}$$

with

$$\text{r.h.s.u} := -A_0 u - B_0(u, u_1) - B_0(u_2, u) + R_0(A_1 d_2, d) + R_0(A_1 d, d_1)$$

and

$$\begin{aligned} \text{r.h.s.d} &:= \lambda_1^{-1} A_1 d - B_1(u, d_1) - B_1(u_2, d) \\ &+ \lambda_1^{-1} (f(d_1) - f(d_2)) + \omega_{Qu} d_1 + \omega_{Qu_2} d \\ &- \lambda_2 \lambda_1^{-1} (A_{Qu} d_1 + A_{Qu_2} d). \end{aligned}$$

Repeated use of (4.16) and arguing as in the proof of Theorem 3.4, owing to $\theta > 0$, $\theta_2 \geq 1$, it is now straightforward to show that

$$\begin{aligned} & \|\text{r.h.s.u}\|_{-\theta-\theta_2} + \|\text{r.h.s.d}\|_{L^2} + \|\sigma_{Qu_1} - \sigma_{Qu_2}\|_{1-\theta-\theta_2} \\ & \leq C \left(\|u\|_{\theta-\theta_2} + \|A_1 d\|_{L^2} \right), \end{aligned}$$

for some constant $C > 0$ which depends on \mathcal{B}_* , but is independent of time. This estimate together with (4.17) and (3.47) gives the desired estimate on the time derivatives in (4.15). The proof is finished. \square

The main result of this section is concerned with the existence of exponential attractors for problem (2.2) in the case $\theta > 0$.

Theorem 4.7. *Let the assumptions of Proposition 4.2 be satisfied. Then the dynamical system $(S_{\theta_2}, \Upsilon_{\beta, s})$ possesses an exponential attractor $\mathcal{M}_{\theta_2, \beta, s} \subset \Upsilon_{\beta, s}$ which is bounded in $\Upsilon_{\beta, s}$. More precisely by definition, we have*

- (a) $\mathcal{M}_{\theta_2, \beta, s}$ is compact and semi-invariant with respect $S_{\theta_2}(t)$, that is,

$$S_{\theta_2}(t)(\mathcal{M}_{\theta_2, \beta, s}) \subseteq \mathcal{M}_{\theta_2, \beta, s}, \quad \forall t \geq 0.$$

- (b) The fractal dimension $\dim_F(\mathcal{M}_{\theta_2, \beta, s}, \mathcal{Y}_{\theta_2})$ of $\mathcal{M}_{\theta_2, \beta, s}$ is finite and an upper bound can be computed explicitly.
- (c) $\mathcal{M}_{\theta_2, \beta, s}$ attracts exponentially fast any bounded subset B of $\Upsilon_{\beta, s}$, that is, there exist a positive nondecreasing function \mathcal{L} and a constant $\tau > 0$ such that

$$\text{dist}_{\mathcal{Y}_{\theta_2}}(S_{\theta_2}(t)B, \mathcal{M}_{\theta_2, \beta, s}) \leq \mathcal{L}(\|B\|_{\Upsilon_{\beta, s}})e^{-\tau t}, \quad \forall t \geq 0.$$

Here $\text{dist}_{\mathcal{Y}_{\theta_2}}$ denotes the Hausdorff semi-distance between sets in \mathcal{Y}_{θ_2} and $\|B\|_{\Upsilon_{\beta, s}}$ stands for the size of B in $\Upsilon_{\beta, s}$. Both \mathcal{L} and τ can be explicitly calculated.

Proof. We apply an abstract result stated in the Appendix, see Proposition 7.4. Recall that by Proposition 4.3, the ball \mathcal{B}_* is absorbing and positively invariant for $S_{\theta_2}(t)$. On the other hand, due to the results proven in this section, we have

$$\sup_{t \geq 0} \|(u(t), d(t))\|_{\Upsilon_{\beta, s}} \leq C_{\beta, s},$$

for every trajectory $\varphi = (u, d)$ originating from $\varphi_0 = (u_0, d_0) \in \mathcal{B}_*$, for some positive constant $C_{\beta, s}$ which is independent of the choice of $\varphi_0 \in \mathcal{B}_*$. We can now apply the abstract result of Proposition 7.4 to the map

$$\mathbb{S} = S_{\theta_2}(T_*) : \mathbb{B} \rightarrow \mathbb{B},$$

where $\mathbb{B} = \mathcal{B}_*$ and $\mathcal{H} = V^{-\theta_2} \times W^1$, for a fixed $T_* > 0$ such that $e^{-\eta T_*} < \frac{1}{2}$, $\eta > 0$ is the same as in Lemma 4.5. To this end, we introduce the functional spaces

$$(4.18) \quad \begin{aligned} \mathcal{V}_1 &:= L^2(0, T; V^{\theta-\theta_2} \times L^2(\Omega)) \cap H^1(0, T; V^{-\theta-\theta_2} \times L^2(\Omega)), \\ \mathcal{V} &:= L^2(0, T; V^{-\theta_2} \times W^1) \end{aligned}$$

and note that \mathcal{V}_1 is compactly embedded into \mathcal{V} due to the Aubin-Lions-Simon compactness lemma. Finally, we introduce the operator $\mathbb{T} : \mathcal{B}_* \rightarrow \mathcal{V}_1$, by $\mathbb{T}\varphi_0 := \varphi \in \mathcal{V}_1$, where φ solves (2.2) with $\varphi(0) = \varphi_0 \in \mathcal{B}_*$. We claim that the maps \mathbb{S}, \mathbb{T} , the spaces $\mathcal{H}, \mathcal{V}, \mathcal{V}_1$ thus defined satisfy all the assumptions of Proposition 7.4. Indeed, the global Lipschitz continuity (7.1) of \mathbb{T} is an immediate corollary of Lemma 4.6, and estimate (7.2) follows from estimate (4.12). Therefore, due to Proposition 7.4, the semigroup $\mathbb{S}(n) = S_{\theta_2}(nT_*)$ generated by the iterations of the operator $\mathbb{S} : \mathcal{B}_* \rightarrow \mathcal{B}_*$ possesses a (discrete) exponential attractor $(\mathcal{M}_{\theta_2, \beta, s})_d$ in \mathcal{B}_* endowed with the topology of $V^{-\theta_2} \times W^1$. In order to construct the exponential attractor $\mathcal{M}_{\theta_2, \beta, s}$ for the semigroup $S_{\theta_2}(t)$ with continuous time, we note that, due to Theorem 3.5, this semigroup is Lipschitz continuous with respect to the initial data in the topology of $V^{-\theta_2} \times W^1$. Moreover, by Lemma 4.4 the map $(t, \varphi_0) \mapsto S_{\theta_2}(t)\varphi_0$ is also uniformly Hölder continuous on $[0, T] \times \mathcal{B}_*$, where \mathcal{B}_* is equipped with the metric topology of $V^{-\theta_2} \times W^1$. Hence, the desired exponential attractor $\mathcal{M}_{\theta_2, \beta, s}$ for the continuous semigroup $S_{\theta_2}(t)$ can be obtained by the standard formula

$$(4.19) \quad \mathcal{M}_{\theta_2, \beta, s} = \bigcup_{t \in [0, T_*]} S_{\theta_2}(t)(\mathcal{M}_{\theta_2, \beta, s})_d.$$

Theorem 4.7 is now proved. \square

As a consequence of the above theorem, we have the following.

Corollary 4.8. *Under the assumptions of Theorem 4.7, there exists a global attractor $\mathcal{A}_{\theta_2, \beta, s}$ which attracts the bounded sets of $\Upsilon_{\beta, s}$. Moreover, $\mathcal{A}_{\theta_2, \beta, s}$ is connected, bounded in $\Upsilon_{\beta, s}$ and $\mathcal{A}_{\theta_2, \beta, s}$ has finite fractal dimension:*

$$\dim_F(\mathcal{A}_{\theta_2, \beta, s}, \mathcal{Y}_{\theta_2}) < \infty.$$

Remark 4.2. In fact due to interpolation, Theorem 4.7 also implies that the fractal dimension of the global and exponential attractors is finite in $V^{\beta-\varepsilon_1} \times W^{2s-\varepsilon_2}$, for every $-\theta_2 < \varepsilon_1 < \beta$ and $n/2 < \varepsilon_2 < 2s$. The attraction property in (c) also holds in the stronger topology of $V^{\beta-\varepsilon_1} \times W^{2s-\varepsilon_2}$.

Note that Proposition 4.2 provides many examples where the conclusion of Theorem 4.7 is satisfied. For example, checking all the requirements of Proposition 4.2 in the case when $\theta = 1$, the conclusions of Theorem 4.7 and Corollary 4.8 are satisfied for the modified 3D Leray-EL- α

(ML-EL- α) model, the 3D SBM-EL model and the 3D NS-EL- α system. These results were not reported anywhere in the literature for the full Ericksen-Leslie model.

5. CONVERGENCE TO STEADY STATES

In this section, we show that any global-in-time bounded energy solution to the full regularized or nonregularized Ericksen-Leslie model (2.2) converges (in a certain sense) to a single equilibrium as time tends to infinity. The proof of the main statements are based on a suitable version of the Łojasiewicz–Simon theorem and the results developed in the previous sections. We emphasize that our subsequent results hold *only* for the energy weak solutions introduced through Definition 2.4, even when uniqueness is not available. In particular, they hold for limit points of the Galerkin approximation scheme exploited in Theorem 3.2, as well as for other approximation schemes in which the energy inequality (2.27) can be proven. Thus, the energy inequality is crucial for investigating the long-time behavior as time goes to infinity. It will also serve as a selection criterion in eliminating all those *non-physical* weak solutions in the framework of Definition 2.3, which may not necessarily satisfy the energy inequality (2.27). Finally, in some cases when the energy solutions become more regular, we can also prove stronger convergence results.

We shall first prove that every energy solution given by Definition 2.4 has a non-empty ω -limit set.

Lemma 5.1. *Let the assumptions of Theorem 3.2 be satisfied, and suppose that g also obeys the following condition:*

$$(5.1) \quad \int_t^\infty \|g(s)\|_{-\theta-\theta_2}^2 ds \lesssim (1+t)^{-(1+\delta)}, \text{ for all } t \geq 0,$$

for some constant $\delta \in (0, 1)$. Let (u, d) be an energy solution in the sense of Definition 2.4. Then, the ω -limit set of (u, d) is nonempty. More precisely, we have

$$(5.2) \quad \lim_{t \rightarrow \infty} u(t) = 0 \text{ weakly in } V^{-\theta_2}$$

and any divergent sequence $\{t_n\} \subset [0, \infty)$ admits a subsequence, denoted by $\{t_{n_k}\}$, such that

$$(5.3) \quad \lim_{t_{n_k} \rightarrow \infty} d(t_{n_k}) = d_* \text{ weakly in } W^1, \text{ strongly in } W^0,$$

for some $d_* \in D(A_1)$ which is a solution of

$$(5.4) \quad A_1 d_* + f(d_*) = 0 \text{ in } \Omega.$$

Proof. First, we recall that an energy solution in the sense of Definition 2.4 exists by virtue of Theorem 3.2. Our proof follows the lines of the argument given in [20, Theorem 2.6] and our arguments developed in Theorem 3.2. We prove our subsequent results in **Case 1** (**Case 2** is analogous and follows with some minor modifications). To this end, let $\{t_n\} \subset [0, \infty)$ be a divergent sequence. The energy inequality (2.27) together with assumption (5.1) implies, at least for a suitable subsequence of $\{t_n\}$, still labelled as $\{t_n\}$, that

$$u(t_n) \rightarrow u_* \text{ weakly in } V^{-\theta_2}, \quad d(t_n) \rightarrow d_* \text{ weakly in } W^1,$$

for some $(u_*, d_*) \in \mathcal{Y}_{\theta_2}$. Consider now the initial value problem (2.2) on the time interval $[t_n, t_{n+1}]$ with the initial values $(u(t_n), d(t_n))$ and observe that $(u_n(t), d_n(t)) := (u(t + t_n), d(t + t_n))$ are also weak solutions of (2.2) for $t \in [0, 1]$. Then, from the energy inequality (2.27) and (2.28), as $t_n \rightarrow \infty$ we infer

$$(5.5) \quad \begin{aligned} u_n &\rightarrow 0 \text{ strongly in } L^2(0, 1; V^{\theta-\theta_2}), \text{ weakly star in } L^\infty(0, 1; V^{-\theta_2}), \\ d_n &\rightarrow \bar{d} \text{ weakly in } L^2(0, 1; W^2), \text{ weakly star in } L^\infty(0, 1; W^1), \end{aligned}$$

for a suitable function \bar{d} . Repeating the comparison arguments developed in the proof of Theorem 3.2, we deduce

$$(5.6) \quad \partial_t d_n \rightarrow \partial_t \bar{d} \text{ weakly in } L^2(0, 1; W^{-2})$$

and, in particular, $\partial_t u_n$ is uniformly bounded in $L^p(0, 1; V^{-\gamma})$ for $p > 1$ and $\gamma \geq 0$ as given by (3.2). In particular, by the Aubin-Lions-Simon compactness criterion and (5.5)-(5.6) we obtain

$$(5.7) \quad d_n \rightarrow \bar{d} \text{ strongly in } L^2(0, 1; W^1) \cap C(0, 1; W^{-2}),$$

as well as

$$(5.8) \quad u_n \rightarrow 0 \text{ strongly in } C(0, 1; V^{-\gamma})$$

for some $\gamma \geq 0$. Moreover, by the definition of Q , one has

$$(5.9) \quad Qu_n \rightarrow 0 \text{ strongly in } L^2(0, 1; V^{\theta+\theta_2}), \text{ weakly star in } L^\infty(0, 1; V^{\theta_2}).$$

Thus, (5.8) yields that $u_* = 0$, which implies (5.2) in view of (5.5) and (5.8). On the other hand, by the energy inequality (2.27) and (2.28) we also have

$$(5.10) \quad \begin{aligned} A_1 d_n + f(d_n) &\rightarrow 0 \text{ strongly in } L^2(0, 1; L^2(\Omega)), \\ A_Q^n d_n &\rightarrow 0 \text{ strongly in } L^2(0, 1; L^2(\Omega)), \\ d_n^T A_Q^n d_n &\rightarrow 0 \text{ strongly in } L^2(0, 1; L^2(\Omega)). \end{aligned}$$

Next, by the estimates (5.5), (5.7)-(5.8), and observing that terms like $(\partial_i Q u_j) d_j$ are a product of functions in $H^{\theta+\theta_2-1}$ and H^1 , and therefore bounded in H^{-2} , for any $\theta, \theta_2 \geq 0$ by Lemma 7.3, we also infer

$$(5.11) \quad \omega_Q^n d_n \rightarrow 0 \text{ strongly in } L^2(0, 1; W^{-2}).$$

Using (5.9), a similar argument entails that

$$(5.12) \quad B_1(u_n, d_n) \rightarrow 0 \text{ strongly in } L^2(0, 1; W^{-2}).$$

Thus, comparing terms in the second equation of (2.2), we also obtain

$$(5.13) \quad \partial_t d_n \rightarrow 0 \text{ strongly in } L^2(0, 1; W^{-2});$$

henceforth, it follows that $\bar{d} = d_*$ for all $t \in [0, 1]$. Letting now $t_n \rightarrow \infty$ in the equation for the director field d_n , satisfying (2.2), we observe that d_* is also a solution of $A_1 d_* + f(d_*) = 0$ in Ω and $d_* \in D(A_1)$, as claimed. Lemma 5.1 is proved. \square

Even though we are dealing with an asymptotically decaying force $g(t)$ due to (5.1), in general we cannot conclude that each energy solution of (2.2) converges to a *single* equilibrium, as the set of steady states associated with (5.4) can be quite complicated (see, e.g., [12, 22]). This means that we are required to prove (5.3) for the whole sequence $\{t_n\}$ and not only a subsequence. The main tool is the same energy functional from (2.15) $E_Q(t) =: \mathcal{E}_Q(u(t), d(t))$, that is,

$$\mathcal{E}_Q(u, d) := \frac{1}{2} \langle u, Qu \rangle + \widehat{\mathcal{E}}(d), \quad \widehat{\mathcal{E}}(d) := \frac{1}{2} \|A_1^{1/2} d\|_{L^2}^2 + \int_{\Omega} W(d) dx.$$

We note that d_* is a critical point of $\widehat{\mathcal{E}}$ over $D(A_1^{1/2})$.

The version of the Łojasiewicz-Simon inequality we need is given by the following lemma, proved in [5, 15].

Lemma 5.2. *There exist constants $\zeta \in (0, 1/2)$ and $C_L > 0$, $\eta > 0$ depending on d_* such that, for any $d \in D(A_1^{1/2})$, if $\|d - d_*\|_1 \leq \eta$, denoting by $\widehat{\mathcal{E}}'$ the Fréchet derivative of $\widehat{\mathcal{E}}$, we have*

$$(5.14) \quad C_L \|\widehat{\mathcal{E}}'(d)\|_{-1} \geq |\widehat{\mathcal{E}}(d) - \widehat{\mathcal{E}}(d_*)|^{1-\zeta}.$$

The following statement is valid for any energy solution (u, d) of Definition 2.4.

Proposition 5.3. *There exists a constant $e_\infty \in \mathbb{R}$ such that $\widehat{\mathcal{E}}(d_*) = e_\infty$, for all solutions d_* satisfying (5.4), and we have*

$$(5.15) \quad \lim_{t \rightarrow \infty} \mathcal{E}_Q(u(t), d(t)) = e_\infty.$$

Moreover, the functional $\Phi(t)$ is nonincreasing along all energy solutions $(u(t), d(t))$ and, for all $t \geq 0$,

$$(5.16) \quad \frac{d}{dt} \Phi(t) \leq - \left(\frac{c_{A_0}}{2} \|u(t)\|_{\theta-\theta_2}^2 - \lambda_1^{-1} \|A_1 d(s) + f(d(s))\|_{L^2}^2 \right),$$

where

$$(5.17) \quad \Phi(t) := \mathcal{E}_Q(u(t), d(t)) + \frac{2\|Q\|_{-\theta_2; \theta_2}^2}{c_{A_0}} \int_t^\infty \|g(s)\|_{-\theta-\theta_2}^2 ds.$$

Our first result is concerned with the convergence of energy solutions of problem (2.2) to single equilibria, showing that their ω -limit set is always a singleton.

Theorem 5.4. *Let the assumptions of Lemma 5.1 hold. The ω -limit set of the component d of any weak energy solution (u, d) of problem (2.2), as given by Definition 2.4, is a singleton. Further we also have*

$$(5.18) \quad \lim_{t \rightarrow \infty} \|d(t) - d_*\|_{L^2} = 0, \quad \lim_{t \rightarrow \infty} \langle u(t), v \rangle = 0, \quad \forall v \in V^{\theta_2}$$

and the following convergence rate:

$$(5.19) \quad \|d(t) - d_*\|_{W^{1,-}} \lesssim (1+t)^{-\chi},$$

for some $\chi \in (0, 1)$ depending on d_* .

Proof. The second claim of (5.18) follows from (5.2). To prove the first claim, we adapt the ideas of [5, 15] and use an argument that we applied in [11, Section 5.3] for a simplified regularized Ericksen-Leslie model. First, we observe that the conclusion of Lemma 5.2 also holds, provided that we choose even a smaller constant $\zeta \in (0, \frac{1}{2}) \cap (0, \delta(1+\delta)^{-1})$, where $\delta < 1$ is the decay rate in (5.1). In order to see that, it suffices to choose a constant $\eta > 0$ in Lemma 5.2 so small that $|\widehat{\mathcal{E}}(d) - \widehat{\mathcal{E}}(d_*)| \leq 1$ whenever $\|d - d_*\|_1 \leq \eta$. Define further

$$\widehat{\Phi}(t) := \mathcal{E}_Q(u(t), d(t)) - \widehat{\mathcal{E}}(d_*) + \frac{2\|Q\|_{-\theta_2; \theta_2}^2}{c_{A_0}} \int_t^\infty \|g(s)\|_{-\theta-\theta_2}^2 ds$$

and notice that $\widehat{\Phi}(t)$ differs from $\Phi(t)$ in (5.17) only by a constant. Hence, setting

$$\Upsilon(t) := \|u(t)\|_{\theta-\theta_2} + \|A_1 d(t) + f(d(t))\|_{L^2},$$

for every $t \geq 0$, from (5.16) we have

$$(5.20) \quad \frac{d}{dt} \widehat{\Phi}(t) \lesssim -\Upsilon^2(t) \leq 0$$

so that $\widehat{\Phi}$ is also a nonincreasing function on $[0, \infty)$. Furthermore, integrating this relation over $(0, \infty)$ and recalling (5.15), we also obtain

$$\int_0^\infty \Upsilon^2(s) ds < \infty,$$

thanks to (5.2)-(5.4). Together with the energy inequality (2.27), then one has

$$(5.21) \quad u \in L^\infty(0, \infty; V^{-\theta_2}) \cap L^2(0, \infty; V^{\theta-\theta_2}),$$

$$(5.22) \quad d \in L^\infty(0, \infty; W^1),$$

$$(5.23) \quad A_1 d + f(d) \in L^2(0, \infty; L^2(\Omega)).$$

As before, these bounds together with proper handling of the other nonlinear terms in the director equation of (2.2) imply

$$(5.24) \quad A_Q d \in L^2(0, \infty; W^{-2}),$$

$$(5.25) \quad \omega_Q d \in L^2(0, \infty; W^{-2}),$$

$$(5.26) \quad B_1(u, d) \in L^2(0, \infty; W^{-2}).$$

In detail the estimates in (5.24)-(5.26) are obtained by application of Lemma 7.3 (Appendix), as follows:

- (1) The terms $(\partial_i Q u_j) d_j$ are a product of functions in $H^{\theta+\theta_2-1}$ and H^1 and therefore bounded in H^{-2} since $\theta, \theta_2 \geq 0$.
- (2) The terms $(Q u_i) \partial_i d_j$ are a product of functions in $H^{\theta+\theta_2}$ and L^2 , and therefore bounded in H^{-2} again, since $\theta, \theta_2 \geq 0$.

Therefore, once again comparing terms in the director equation for d from (2.2), the estimates (5.23)-(5.26) entail

$$(5.27) \quad \partial_t d \in L^2(0, \infty; W^{-2}).$$

Our next goal is to show that there exists $t_0 > 0$ sufficiently large, such that $\partial_t d \in L^1(t_0, \infty; W^{-2})$. Now, define

$$\Sigma := \{t \geq 1 : \|d(t) - d_*\|_{L^2} \leq \eta/3\}$$

and observe that Σ is unbounded by Lemma 5.1. For every $t \in \Sigma$, we define

$$\tau(t) = \sup\{t' \geq t : \sup_{s \in [t, t']} \|d(s) - d_*\|_{L^2} < \eta\}.$$

By continuity, $\tau(t) > t$ for every $t \in \Sigma$. Let now $t_0 \in \Sigma$ and divide the interval $J := [t_0, \tau(t_0))$ into two subsets

$$\Sigma_1 := \left\{ t \in J : \Upsilon(t) \geq \left(\int_t^{\tau(t_0)} \|g(s)\|_{-\theta-\theta_2}^2 ds \right)^{1-\zeta} \right\}, \quad \Sigma_2 := J \setminus \Sigma_1.$$

Setting further, as above,

$$\widehat{\Phi}(t) := \mathcal{E}_Q(u(t), d(t)) - \widehat{\mathcal{E}}(d_*) + \frac{2\|Q\|_{-\theta_2; \theta_2}^2}{c_{A_0}} \int_t^{\tau(t_0)} \|g(s)\|_{-\theta-\theta_2}^2 ds$$

we notice that $\widehat{\Phi}(t)$ again satisfies (5.20) for every $t \in J$, and $\widehat{\Phi}$ is a nonincreasing function on J . Moreover, for every $t \in J$ we have

$$(5.28) \quad \begin{aligned} \frac{d}{dt} \left(|\widehat{\Phi}(t)|^\zeta \operatorname{sgn}(\widehat{\Phi}(t)) \right) &= \zeta |\widehat{\Phi}(t)|^{\zeta-1} \frac{d}{dt} \widehat{\Phi}(t) \\ &\lesssim -|\widehat{\Phi}(t)|^{\zeta-1} \Upsilon^2(t), \end{aligned}$$

which implies that the functional $\operatorname{sgn}(\widehat{\Phi}(t))|\widehat{\Phi}(t)|^\zeta$ is decreasing on J . By (5.14) and Proposition 5.3, for every $t \in \Sigma_1$ we can easily establish

$$(5.29) \quad \begin{aligned} |\widehat{\Phi}(t)|^{1-\zeta} &\leq \left| \mathcal{E}_Q(u(t), d(t)) - \widehat{\mathcal{E}}(d_*) \right|^{1-\zeta} + \left(\frac{2\|Q\|_{-\theta_2; \theta_2}^2}{c_{A_0}} \int_t^{\tau(t_0)} \|g(s)\|_{-\theta-\theta_2}^2 ds \right)^{1-\zeta} \\ &\lesssim \Upsilon(t), \end{aligned}$$

owing to the basic inequality

$$\|u(t)\|_{-\theta_2}^2 \lesssim \|u(t)\|_{\theta-\theta_2}^{\frac{1}{1-\zeta}}, \text{ for a.e. } t > 0,$$

which holds thanks to the embedding $V^{\theta-\theta_2} \subseteq V^{-\theta_2}$ and (5.21). Combining now (5.29) with (5.28) yields

$$(5.30) \quad -\frac{d}{dt} \left(|\widehat{\Phi}(t)|^\zeta \operatorname{sgn}(\widehat{\Phi}(t)) \right) \gtrsim \Upsilon(t).$$

Moreover, exploiting (5.30) we have

$$(5.31) \quad \begin{aligned} \int_{\Sigma_1} \Upsilon(s) ds &\lesssim - \int_{\Sigma_1} \frac{d}{ds} \left(|\widehat{\Phi}(s)|^\zeta \operatorname{sgn}(\widehat{\Phi}(s)) \right) ds \\ &\lesssim \left(|\widehat{\Phi}(t_0)|^\zeta + |\widehat{\Phi}(\tau(t_0))|^\zeta \right) < \infty, \end{aligned}$$

where we interpret the term involving $\tau(t_0)$ on the right hand side of (5.31) as 0 if $\tau(t_0) = \infty$ (recall (5.15)). On the other hand, if $t \in \Sigma_2$, using assumption (5.1) we obtain

$$(5.32) \quad \Upsilon(t) \leq \left(\int_t^{\tau(t_0)} \|g(s)\|_{-\theta-\theta_2}^2 ds \right)^{1-\zeta} \lesssim (1+t)^{-(1-\zeta)(1+\delta)},$$

so once again the function Υ is dominated by an integrable function on Σ_2 since $\zeta(1+\delta) < \delta$. Combining the inequalities (5.31), (5.32), we deduce that Υ is absolutely integrable on J and

$$(5.33) \quad \lim_{t_0 \rightarrow \infty, t_0 \in \Sigma} \int_{t_0}^{\tau(t_0)} \Upsilon(s) ds = 0.$$

On the other hand, recalling estimates (5.21)-(5.26) and the observations (1)-(2), from the second equation of (2.2) it follows that

$$(5.34) \quad \begin{aligned} \|\partial_t d(t)\|_{-2} &\lesssim \|B_1(u(t), d(t))\|_{L^2} + \|A_1 d(t) + f(d(t))\|_{L^2} \\ &\quad + \|A_Q d(t)\|_{L^2} + \|\omega_Q d(t)\|_{L^2} \\ &\lesssim \|u(t)\|_{\theta-\theta_2} \|d(t)\|_1 + \|A_1 \phi(t) + f(\phi(t))\|_{L^2} \\ &\lesssim \Upsilon(t). \end{aligned}$$

Consequently, we also have

$$(5.35) \quad \lim_{t_0 \rightarrow \infty, t_0 \in \Sigma} \int_{t_0}^{\tau(t_0)} \|\partial_t d(s)\|_{-2} ds = 0.$$

This fact, combined with a simple contradiction argument (see [5], [11]) yields that we must have $\tau(t_0) = \infty$, for some sufficiently large $t_0 \in \Sigma$. Thus, $\partial_t d \in L^1(t_0, \infty; W^{-2})$ as desired. By compactness and a basic interpolation inequality, we have $d(t) \rightarrow d_*$ in the strong topology of W^{1-} . Hence, the ω -limit set of the d component of any weak energy solution (u, d) is the singleton d_* , which is a solution of (5.4). The estimate of the rate of convergence in (5.19) is a straightforward consequence of (5.28)-(5.29), the definition of Φ and basic interpolation results. We leave the details to the interested reader. The proof of Theorem 5.4 is complete. \square

Remark 5.1. Our theorem covers all the special cases listed in Table 1 and many other models (see Remark 3.1). In particular, our result yields convergence to a single steady state $(0, d_*)$ of any weak *energy solution* of the full three dimensional NSE-EL, Leray-EL- α , ML-EL- α , NSV-EL, SBM-EL, NS-EL- α models. None of these results have been reported previously.

We can derive a sufficient condition such that a stronger convergence result holds in (5.18).

Theorem 5.5. *Let $(u, d) \in L^\infty(0, \infty; \mathcal{Y}_{\theta_2})$ be an energy solution in the sense of Lemma 5.1, determined by the assumptions of Theorem 3.2. In addition, assume*

$$(5.36) \quad \theta + \theta_2 \geq 1 \text{ and } d \in L^\infty(0, \infty; L^\infty(\Omega)).$$

Then, there holds

$$(5.37) \quad \lim_{t \rightarrow +\infty} \left(\|u(t)\|_{-\theta_2} + \|A_1^{1/2}(d(t) - d_*)\|_{L^2} \right) = 0,$$

where $d_ \in D(A_1)$ is a solution of (5.4).*

Proof. We recall that each energy solution (u, d) of Definition 2.4 satisfies the bounds (5.21)-(5.23) and that $y \in L^1(0, \infty)$, owing to $V^{\theta-\theta_2} \subseteq V^{-\theta_2}$, where we have set

$$y(t) := \frac{1}{2} \langle u, Qu \rangle + \|\rho(t)\|_{-1}^2, \quad \rho(t) := A_1 d(t) + f(d(t)).$$

In particular, by (2.25)-(2.28) we recall that

$$(5.38) \quad \int_0^\infty \left(\|u(s)\|_{-\theta_2}^2 + \|\rho(s)\|_{L^2}^2 + \mu_1 \|d^T A_Q d\|_{L^2}^2 \right) ds < \infty.$$

By the second condition in (5.36), there exists a constant $M > 0$ independent of time such that

$$\|d\|_{L^\infty(\mathbb{R}_+; L^\infty(\Omega))} \leq M.$$

Our goal is to show that y satisfies the inequality

$$(5.39) \quad \frac{dy}{dt}(t) \leq C + \Lambda(t), \text{ for all } t \geq 0,$$

for some constant $C > 0$ independent of time and some function $\Lambda \in L^1(0, \infty)$. Then, the application of [23, Lemma 6.2.1] yields $y(t) \rightarrow 0$ as $t \rightarrow \infty$; the convergence (5.37) is then an immediate consequence of this crucial fact, owing to the basic inequality

$$\|A_1^{1/2}(d(t) - d_*)\|_{L^2} \lesssim \|\rho(t)\|_{-1} + \|f(d) - f(d_*)\|_{-1} + \|A_1 d_* + f(d_*)\|_{-1}$$

and (5.18), (5.4). Of course, (5.39) can be justified by employing a proper approximation scheme, such as the one used in the proof of Theorem 3.2.

To this end, we first pair the first equation of (2.2) with Qu , then use assumption (ii) of Theorem 3.2, to deduce the identity

$$(5.40) \quad \frac{1}{2} \frac{d}{dt} \langle u, Qu \rangle + \langle A_0 u, Qu \rangle = \langle g, Qu \rangle + \langle R_0(\rho, d), Qu \rangle + \langle \sigma_Q, \nabla(Qu) \rangle,$$

where the tensor σ_Q is given by (1.5). On the other hand, in view of the second equation of (2.2) we have

$$(5.41) \quad \begin{aligned} \frac{d}{dt} \|\rho\|_{-1}^2 &= -\langle B_1(u, d), \rho \rangle + \langle \omega_Q d, \rho \rangle - \frac{\lambda_2}{\lambda_1} \langle A_Q d, \rho \rangle + \frac{1}{\lambda_1} \|\rho\|_{L^2}^2 \\ &\quad - \left\langle f'(d) B_1(u, d), A_1^{-1} \rho \right\rangle + \left\langle f'(d) \omega_Q d, A_1^{-1} \rho \right\rangle \end{aligned}$$

$$-\frac{\lambda_2}{\lambda_1} \left\langle f'(d) A_Q d, A_1^{-1} \rho \right\rangle + \frac{1}{\lambda_1} \left\langle f'(d) \rho, A_1^{-1} \rho \right\rangle.$$

Adding the relations (5.40)-(5.41) together, then using (2.7) and noting that $\langle B_1(u, d), \rho \rangle = \langle R_0(\rho, d), Qu \rangle$, one has

$$(5.42) \quad \begin{aligned} & \frac{dy}{dt} + c_{A_0} \|u\|_{\theta-\theta_2}^2 - \frac{1}{\lambda_1} \|\rho\|_{L^2}^2 \\ & \leq \langle g, Qu \rangle + \langle \sigma_Q, \nabla(Qu) \rangle + \langle \omega_Q d, \rho \rangle - \frac{\lambda_2}{\lambda_1} \langle A_Q d, \rho \rangle \\ & \quad - \left\langle f'(d) B_1(u, d), A_1^{-1} \rho \right\rangle + \left\langle f'(d) \omega_Q d, A_1^{-1} \rho \right\rangle \\ & \quad - \frac{\lambda_2}{\lambda_1} \left\langle f'(d) A_Q d, A_1^{-1} \rho \right\rangle + \frac{1}{\lambda_1} \left\langle f'(d) \rho, A_1^{-1} \rho \right\rangle \\ & =: I_1 + \dots + I_8. \end{aligned}$$

We now obtain proper bounds for the terms on the right-hand side in the following manner:

(b1) As usual for the first one, for every $\delta > 0$ we have

$$|I_1| \leq \delta \|u\|_{\theta-\theta_2}^2 + C_\delta \|g\|_{-\theta-\theta_2}^2.$$

(b2) For I_3, I_4 , exploiting (5.36) together with the Sobolev embedding $V^{\theta-\theta_2} \subseteq V^{1-2\theta_2}$, as $\theta + \theta_2 \geq 1$, yields

$$|I_3| + |I_4| \leq \|\rho\|_{L^2} \|\nabla(Qu)\|_{L^2} \|d\|_{L^\infty} \leq \delta \|u\|_{\theta-\theta_2}^2 + C_{\delta, M} \|\rho\|_{L^2}^2.$$

(b3) Using the definition for σ_Q , we further split the second term I_2 into three more terms I_{21}, I_{22}, I_{23} . The first one we bound as follows:

$$\begin{aligned} |I_{21}| &= \langle \mu_1(d^T A_Q d) d \otimes d, \nabla Qu \rangle \\ &\leq \mu_1 \|\nabla Qu\|_{L^2} \|d^T A_Q d\|_{L^2} \|d \otimes d\|_{L^\infty} \\ &\lesssim \mu_1 \|u\|_{\theta-\theta_2} \|d^T A_Q d\|_{L^2} \|d\|_{L^\infty}^2 \\ &\leq \delta \|u\|_{\theta-\theta_2}^2 + C_{\delta, M} (\mu_1 \|d^T A_Q d\|_{L^2})^2. \end{aligned}$$

Next, since by definition $\mathcal{N}_Q = \lambda_1^{-1} \rho - \lambda_2/\lambda_1 A_Q d$, proceeding as for the estimate for I_3 , we get

$$\begin{aligned} |I_{22}| &= |\langle \mu_2 \mathcal{N}_Q \otimes d + \mu_3 d \otimes \mathcal{N}_Q, \nabla Qu \rangle| \\ &\lesssim \|\nabla Qu\|_{L^2} \|\mathcal{N}_Q\|_{L^2} \|d\|_{L^\infty} \\ &\leq C_M \|u\|_{\theta-\theta_2} (\|\rho\|_{L^2} + \|A_Q d\|_{L^2}) \\ &\leq C_M \|u\|_{\theta-\theta_2} \|\rho\|_{L^2} + C_M \|u\|_{\theta-\theta_2}^2, \end{aligned}$$

owing once more to the boundedness of d . Finally, a similar argument gives the same bound:

$$\begin{aligned} |I_{23}| &= |\langle \mu_5(A_Q d) \otimes d + \mu_6 d \otimes (A_Q d), \nabla Qu \rangle| \\ &\leq C_M \|u\|_{\theta-\theta_2} (\|\rho\|_{L^2} + 1) + C_M \|u\|_{\theta-\theta_2}^2. \end{aligned}$$

(b4) Since the term $f'(d)(B_1(u, d))$ is bounded in W^{-1} as a product of vector-valued functions in W^1 and L^2 , using $V^{\theta-\theta_2} \subseteq V^{-\theta_2}$ we derive

$$\begin{aligned} |I_3| &= \left| \left\langle f'(d) B_1(u, d), A_1^{-1} \rho \right\rangle \right| \\ &\lesssim \|f'(d)\|_1 \|B_1(u, d)\|_{L^2} \|\rho\|_{-1} \\ &\lesssim \|f'(d)\|_1 \|\nabla Qu\|_1 \|A_1 d\|_{L^2} \|\rho\|_{-1} \\ &\leq \delta \|u\|_{\theta-\theta_2}^2 + C_\delta (\|\rho\|_{L^2} + \|f(d)\|_{L^2})^2 \|f'(d)\|_1^2 \|\rho\|_{-1}^2. \end{aligned}$$

(b5) Deriving a bound for I_4 and I_5 , one argues exactly in the same fashion using the definition of the tensors A_Q and ω_Q . One has

$$|I_4| = \left| \left\langle f'(d) \omega_Q d, A_1^{-1} \rho \right\rangle \right|$$

$$\begin{aligned}
&\lesssim \left\| f'(d) \right\|_1 \|\omega_Q d\|_{L^2} \|\rho\|_{-1} \\
&\lesssim \left\| f'(d) \right\|_1 \|\nabla Q u\|_{L^2} \|d\|_{L^\infty} \|\rho\|_{-1} \\
&\leq \delta \|u\|_{\theta-\theta_2}^2 + C_{\delta,M} \left\| f'(d) \right\|_1^2 \|\rho\|_{-1}^2.
\end{aligned}$$

We also get the same upper bound for I_5 .

(b6) For the final term I_2 , we use the fact that $f'(d)\rho$ is bounded in W^{-1} as a product of vector-valued functions in $W^1 \times L^2$. One readily obtains the estimate:

$$|I_8| = \frac{1}{\lambda_1} \left| \left\langle f'(d)\rho, A_1^{-1}\rho \right\rangle \right| \leq \delta \|\rho\|_{L^2}^2 + C_\delta \left\| f'(d) \right\|_1^2 \|\rho\|_{-1}^2.$$

Collect now all the previous estimates from (b1)-(b6) and insert them on the right-hand side of (5.42). Choosing a sufficiently small $\delta \sim \min(c_{A_0}, -\lambda_1^{-1}) > 0$, taking into account the uniform in-time bounds $\|\rho(t)\|_{-1} \leq C$, $\|f'(d(t))\|_1 \leq C$ and $\|(u(t), d(t))\|_{\mathcal{V}_{\theta_2}} \leq C$, which hold for all times $t \geq 0$, and the basic controls (5.38), (5.1), we readily infer the validity of inequality (5.39) for a proper function $\Lambda \in L^1(0, \infty)$. Hence, we have proved our claim and the proof is finished. \square

Remark 5.2. The second assumption of (5.36) can be slightly weakened if $\mu_1 = 0$ ($\lambda_2 \neq 0$). We recall that when $\lambda_2 = 0$, $\mu_1 \geq 0$, the second of (5.36) is already satisfied on account of Proposition 3.1.

For example, setting $\theta = 1$, $\theta_1 = \theta_2 = 0$ and $\mathcal{V} = \mathcal{V}_{per}$, $\Omega = \mathbb{T}^2$, global existence of a unique strong solution in the class

$$(5.43) \quad (u, d) \in L^\infty(0, \infty; V^1 \times D(A_1))$$

for the two dimensional 2D Navier-Stokes-Ericksen-Leslie model was established in [22, Lemma 4.3 and Remark 4] in either of the following cases (a) $\mu_1 = 0$, $\lambda_2 \neq 0$, (b) $\mu_1 \geq 0$, $\lambda_2 = 0$. We observe that due to the additional regularity (5.43) and the embedding $D(A_1) \subset L^\infty$, the second of (5.36) is automatically satisfied for this model and so the conclusion of Theorem 5.5 holds. In this case, the convergence result can also be found in [22, Theorem 4.2]. On the other hand, the convergence to a single steady state $(0, d_*)$ of any (regular) energy solution, satisfying the assumptions of Theorem 3.3 is also ensured on account of Remark 3.7 and Proposition 4.2. In particular, this is true for the 3D modified Leray-EL- α (ML-EL- α) model, the 3D SBM-EL model and the 3D NS-EL- α system in the general case when $\mu_1 \geq 0$ and $\lambda_2 \neq 0$. Besides, in the case (b) above, the 3D NSV-EL model possesses (unique) energy solutions that converge to single steady states as concluded by (5.37). These convergence results were not previously reported in the literature for any of these models.

6. CONCLUDING REMARKS

In this article, we consider a general family of regularized Ericksen-Leslie models which captures some specifics and variants of the models that have not been considered or analyzed anywhere in the literature before. We give a unified analysis of the Ericksen-Leslie system using tools in nonlinear analysis and Sobolev function theory together with energy methods, and then use them to obtain sharp results. In particular, in Section 3 we develop some well-posedness results for our family of nonlinear models, which include existence results (Section 3.1), regularity results (Section 3.2), and uniqueness and stability results (Section 3.3). In Section 4, we show the existence of a finite-dimensional global attractor in the case $\theta > 0$ and give some further properties, by first establishing the existence of an exponential attractor. In Section 5, we prove the asymptotic stabilization as time goes to infinity of any energy solution for our problem (2.2) to a single steady state. The present unified analysis can be exploited to extend and establish existence, regularity and existence of finite dimensional attractor results also in the case $\theta = 0$; this case is more delicate and requires a more refined analysis which lies beyond the scope of the present article. Indeed, problem (2.2) with $\theta = 0$ can be seen as a non-dissipative system in which the fluid equation loses its parabolic character and behaves more like a hyperbolic equation. For instance, this is the case when the velocity component satisfies the 3D Navier-Stokes-Voigt equation. For a simplified regularized Ericksen-Leslie model ($\sigma_Q \equiv 0$, $\omega_Q \equiv 0$ and $\lambda_2 = 0$), such results have already been established in [11]. For the full regularized Ericksen-Leslie model (2.2), we will consider such questions in a forthcoming contribution.

We conclude this section with some remarks on the assumption about the space \mathcal{V} in (2.16), and the precise connections between the models as introduced in Table 2 and their equivalent formulations which are most recognizable in the physics literature. To this end, let us assume that the fluid velocity satisfies either one of the following regularized versions of the 3D Navier-Stokes equations ($M = I$, $Q = I$, $A_0 = -\mu_4\Delta$):

- (1) The 3D Leray- α system with $\theta = 1$, $\theta_1 = 1$, $\theta_2 = 0$:

$$(6.1) \quad \partial_t u - \mu_4 \Delta u + (\Pi u \cdot \nabla) u + \nabla p = \vec{F}(d),$$

with $\Pi = (I - \alpha^2 \Delta)^{-1}$ and $Q = I$. In this case, $u = Q^{-1}v (= v)$ is the fluid velocity as in (1.1).

- (2) The modified 3D Leray- α (ML- α) system with $\theta = 1$, $\theta_1 = 0$, $\theta_2 = 1$:

$$(6.2) \quad \partial_t (v - \alpha^2 \Delta v) - \mu_4 \Delta (v - \alpha^2 \Delta v) + (v - \alpha^2 \Delta v) \cdot \nabla v + \nabla p = \vec{F}(d),$$

with $Q = \Pi$ and $M = I$. In this case, $v = \Pi u$ is the (regularized) fluid velocity as in (1.1).

- (3) The 3D simplified Bardina model with $\theta = \theta_1 = \theta_2 = 1$:

$$(6.3) \quad \partial_t (v - \alpha^2 \Delta v) - \mu_4 \Delta (v - \alpha^2 \Delta v) + v \cdot \nabla v + \nabla p = \vec{F}(d),$$

with $Q = \Pi$ and $M = \Pi$. As above in (3), $v = \Pi u$ is the (regularized) fluid velocity for the system (1.1).

- (4) The 3D Navier-Stokes-Voigt equation with $\theta = 0$, $\theta_1 = \theta_2 = 1$:

$$(6.4) \quad \partial_t (v - \alpha^2 \Delta v) - \mu_4 \Delta v + v \cdot \nabla v + \nabla p = \vec{F}(d),$$

with $Q = M = \Pi$. Again $v = \Pi u$ (i.e., $u = \Pi^{-1}v$) corresponds to the (regularized) fluid velocity.

- (5) The 3D Lagrangian averaged Navier-Stokes- α equation with $\theta = 1$, $\theta_1 = 0$, $\theta_2 = 1$:

$$(6.5) \quad \partial_t (v - \alpha^2 \Delta v) - \mu_4 \Delta (v - \alpha^2 \Delta v) + v \cdot \nabla (v - \alpha^2 \Delta v) + \nabla (v^T) \cdot (v - \alpha^2 \Delta v) + \nabla p = \vec{F}(d),$$

with $M = \Pi$ and $Q = I$. In this case $v = Q^{-1}u = u$ corresponds to the (regularized) fluid velocity.

Above in (1)-(5), the flow is incompressible (i.e., $\operatorname{div}(v) = 0$), p denotes pressure and \vec{F} consists of a body force $g(t)$ acting on the fluid as well as the stresses/forces due to the coupling of the fluid velocity with the director field d , i.e.,

$$\vec{F}(d) \stackrel{\text{def}}{=} A_1 d \cdot \nabla d + \operatorname{div}(\sigma_Q) + g,$$

such that d obeys the equation

$$(6.6) \quad \partial_t d + v \cdot \nabla d - \omega_Q d + \frac{\lambda_2}{\lambda_1} A_Q d = \frac{1}{\lambda_1} (A_1 d + \nabla_d W(d)).$$

Here, A_Q, ω_Q, σ_Q are given in (1.3) and (1.5), respectively. In particular, we have $A_Q = (\nabla v + \nabla^T v)/2$ and $\omega_Q = (\nabla v - \nabla^T v)/2$.

We emphasize that problem (1.1)-(1.2), for any of the choices of the parameters $(\theta, \theta_1, \theta_2)$ in (1)-(5) above, is in fact equivalent to any regularized 3D Ericksen-Leslie model (oREL) in which the fluid velocity v satisfies either one of the equations (6.1)-(6.5) above and the director field d satisfies (6.6). Indeed this is the case when the operators $Q : V^{-\theta_2} \rightarrow V^{\theta_2}$ and $Q^{-1} : V^{\theta_2} \rightarrow V^{-\theta_2}$ are isometries. Furthermore, according to the statements proven in Section 3, the transformed problem (1.1)-(1.2) is well posed in $V^\beta \times W^l$ for some $\beta \geq -\theta_2$, $l \geq 1$; this makes any of the (oREL) problems for (6.1)-(6.5), (6.6) well-posed in $V^{\beta+2\theta_2} \times W^l$, thus generating a solution semigroup of operators

$$\begin{aligned} \widehat{S}_{\theta_2}(t) : V^{\beta+2\theta_2} \times W^l &\rightarrow V^{\beta+2\theta_2} \times W^l, \\ (v_0, d_0) &\mapsto (v(t), d(t)). \end{aligned}$$

More precisely, if the system (1.1)-(1.2) generates a semigroup of solution operators S_{θ_2} , as determined by the conditions of Section 3,

$$S_{\theta_2}(t) : V^\beta \times W^l \rightarrow V^\beta \times W^l, \quad (u_0, d_0) \mapsto (u(t), d(t))$$

then this semigroup is linked through the corresponding semigroup $\widehat{S}_{\theta_2}(t)$ of any of the (oREL) problems above by the relation

$$u(t) = Q^{-1}v(t), \quad \forall t \geq 0.$$

Concerning the longtime behavior of \widehat{S}_{θ_2} , then \widehat{S}_{θ_2} possesses a global attractor $\widehat{\mathcal{A}}$, which can be seen to satisfy

$$\widehat{\mathcal{A}} = \{(v, d) \in V^{\beta+2\theta_2} \times W^l : v = Qu, (u, d) \in \mathcal{A}\},$$

where \mathcal{A} is a global attractor associated with any dynamical system for the solution operator S_{θ_2} .

Finally, our last comment is about assumption (2.16). For this, let us now consider Ω as a compact Riemannian manifold with boundary Γ and take again $E = T\Omega$ the tangent bundle. We observe that the assumption on \mathcal{V} in (2.16) is satisfied in a more general setting than suggested by the example given in Section 2.2. Indeed, in the context of the specified regularized models of (1)-(5), it suffices to consider \mathcal{V} as a closed subspace of

$$\mathcal{V}_{\text{ns}} = \{v \in C^\infty(T\Omega) : \operatorname{div}(v) = 0, v \cdot n = 0 \text{ on } \Gamma\}.$$

In this case, (2.16) is clearly satisfied by the velocity $v \in \mathcal{V} \subseteq \mathcal{V}_{\text{ns}}$ of any of the problems (1)-(5).

7. APPENDIX

In this section, we include some supporting material on Grönwall-type inequalities, Sobolev inequalities and abstract results. The first lemma is a slight generalization of the usual Grönwall-type inequality.

Lemma 7.1. *Let $\mathcal{E} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\mathcal{E}(t) + 2\eta\mathcal{E}(t) \leq h(t)\mathcal{E}(t) + l(t) + k,$$

where $\eta > 0$, $k \geq 0$ and $\int_s^t h(\tau) d\tau \leq \eta(t-s) + m$, for all $t \geq s \geq 0$ and some $m \in \mathbb{R}$, and $\int_t^{t+1} l(\tau) d\tau \leq \gamma < \infty$. Then, for all $t \geq 0$,

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^m e^{-\eta t} + \frac{2\gamma e^{m+\eta}}{e^\eta - 1} + \frac{ke^m}{\eta}.$$

With $s, p \in \mathbb{R}_+$, let $W^{s,p}$ be the standard Sobolev space on an n -dimensional compact Riemannian manifold with $n \geq 2$. The following result states the classical Gagliardo-Nirenberg-Sobolev inequality (cf. [1, 13] and [6, 7]).

Lemma 7.2. *Let $0 \leq k < m$ with $k, m \in \mathbb{N}$ and numbers $p, q, q \in [1, \infty]$ satisfy*

$$k - \frac{n}{p} = \tau \left(m - \frac{n}{q} \right) - (1 - \tau) \frac{n}{r}.$$

Then there exists a positive constant C independent of u such that

$$\|u\|_{W^{k,p}} \leq C \|u\|_{W^{m,q}}^\tau \|u\|_{L^r}^{1-\tau},$$

with $\tau \in [\frac{k}{m}, 1]$ provided that $m - k - \frac{n}{r} \notin \mathbb{N}_0$, and $\tau = \frac{k}{m}$ provided that $m - k - \frac{n}{r} \in \mathbb{N}_0$.

We state here a standard result on pointwise multiplication of functions in the Sobolev spaces $H^k = W^{k,2}$ (see [18]; cf. also [14]).

Lemma 7.3. *Let s, s_1 , and s_2 be real numbers satisfying*

$$s_1 + s_2 \geq 0, \quad \min(s_1, s_2) \geq s, \quad \text{and} \quad s_1 + s_2 - s > \frac{n}{2},$$

where the strictness of the last two inequalities can be interchanged if $s \in \mathbb{N}_0$. Then, the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$H^{s_1} \otimes H^{s_2} \rightarrow H^s.$$

Our construction of an exponential attractor is based on the following abstract result [10, Proposition 4.1].

Proposition 7.4. *Let $\mathcal{H}, \mathcal{V}, \mathcal{V}_1$ be Banach spaces such that the embedding $\mathcal{V}_1 \hookrightarrow \mathcal{V}$ is compact. Let \mathbb{B} be a closed bounded subset of \mathcal{H} and let $\mathbb{S} : \mathbb{B} \rightarrow \mathbb{B}$ be a map. Assume also that there exists a uniformly Lipschitz continuous map $\mathbb{T} : \mathbb{B} \rightarrow \mathcal{V}_1$, i.e.,*

$$(7.1) \quad \|\mathbb{T}b_1 - \mathbb{T}b_2\|_{\mathcal{V}_1} \leq L \|b_1 - b_2\|_{\mathcal{H}}, \quad \forall b_1, b_2 \in \mathbb{B},$$

for some $L \geq 0$, such that

$$(7.2) \quad \|\mathbb{S}b_1 - \mathbb{S}b_2\|_{\mathcal{H}} \leq \gamma \|b_1 - b_2\|_{\mathcal{H}} + K \|\mathbb{T}b_1 - \mathbb{T}b_2\|_{\mathcal{V}}, \quad \forall b_1, b_2 \in \mathbb{B},$$

for some constant $0 \leq \gamma < \frac{1}{2}$ and $K \geq 0$. Then, there exists a (discrete) exponential attractor $\mathcal{M}_d \subset \mathbb{B}$ of the semigroup $\{\mathbb{S}(n) := \mathbb{S}^n, n \in \mathbb{Z}_+\}$ with discrete time in the phase space \mathcal{H} , which satisfies the following properties:

- *semi-invariance:* $\mathbb{S}(\mathcal{M}_d) \subset \mathcal{M}_d$;
- *compactness:* \mathcal{M}_d is compact in \mathcal{H} ;
- *exponential attraction:* $\text{dist}_{\mathcal{H}}(\mathbb{S}^n \mathbb{B}, \mathcal{M}_d) \leq C_0 e^{-\chi n}$, for all $n \in \mathbb{N}$ and for some $\chi > 0$ and $C_0 \geq 0$, where $\text{dist}_{\mathcal{H}}$ denotes the standard Hausdorff semidistance between sets in \mathcal{H} ;
- *finite-dimensionality:* \mathcal{M}_d has finite fractal dimension in \mathcal{H} .

Moreover, the constants C_0 and χ , and the fractal dimension of \mathcal{M}_d can be explicitly expressed in terms of L , K , γ , $\|\mathbb{B}\|_{\mathcal{H}}$ and Kolmogorov's κ -entropy of the compact embedding $\mathcal{V}_1 \hookrightarrow \mathcal{V}$, for some $\kappa = \kappa(L, K, \gamma)$. We recall that the Kolmogorov κ -entropy of the compact embedding $\mathcal{V}_1 \hookrightarrow \mathcal{V}$ is the logarithm of the minimum number of balls of radius κ in \mathcal{V} necessary to cover the unit ball of \mathcal{V}_1 .

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